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OPTIMIZATION OF AERIAL DEFENSE PATROLS FOR DETECTION OF AERIAL TARGETS

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FOREWORD

This research paper was prepared for the Applied Mathematics Research Laboratory, Aerospace Research Laboratories, Office of Aerospace Research by Mr. Gwon H. Lum and Mrs. Mary D. Lum under Task 707101, Research in Mathematical Statistics and Probability, of Project 7071, Mathematical Techniques of Aerodynamics.

Since Mr. Lum is affiliated with the Directorate of Synthesis, Deputy of Studies and Analysis, Systems Engineering Group, Research and Technology Division of the Air Force Systems Command, this report represents a joint internal effort of AFSC and OAR. The problem originated in AFSC, arising from an operations research study on the feasibility of an air command post for counter insurgency (COIN) operations, reported in ASD TDR 63-292 "Counter-Insurgenc Study," Volume 6: "Patrol and Encounter", by G. H. Lum, C. R. Poli, and M. I Lum, July 1963 (SECRET). The question of effectiveness in detecting aerial targets by aircraft with simple pattern movements motivated the probabilistic formulation contained in this report.

The authors gratefully acknowledge the numerous helpful suggestions and cr icisms of Mr. Robert N. Orpett in putting the problem into proper perspective v regard to physical aspects. Thanks are also due to Lt. Col. John V. Armitage, Dr. H. Leon Harter, and Dr. Paul R. Rider for reading the manuscript, Miss Eva Brandenburg for typing, and Mr. Leonard Stark for drawing the figures.

ABSTRACT

A mathematical representation, with detection probability used as a measure of detection capability, is formulated for constant velocity moving-target-by-moving patrol detection where the patrol employs simple pattern movements. The most essential idea pervading the theory is that the space-time target region separates into two non-overlapping subregions characterized as follows: those target points which can be detected at least once ("detection region") and those target points which can never be detected by the patrol. This formulation leads to a simple mechanism for determining whether the patrol starting at some arbitrary point in the pattern will or will not detect a target starting from an arbitrary point in the plane of the pattern. Also whenever detection is possible, the number of times the target is detected and the corresponding durations are theoretically determinable, though no attempt at these calculations have been made in this report.

Although the area of radar coverage of the patrol is taken to be circular in this report, to correspond to conditions attainable in practice, the shape of radar coverage is easily (in theory) generalizeable to any form. The basic concepts, introduced in Section 2, hold for all patterns having a finite number of straight-line segments, including rectangular and cross-over patterns. However, for simplicity, we have restricted the discussion in this report to the back-and-forth pattern. The basic concepts are: effective patrol length, full-cycle target region, duality of relative motion, and detection region.

In Section 3, we calculate the detection probabilities for the special case where the target path is perpendicular to the patrol path, a situation for which the detection probabilities are at a minimum. This situation is referred to as the "worst case".

Therefore the calculated numbers are lower-bound values of detection probabilities for general target headings (other then perpendicular).

Finally (Section 4) for given values of target-to-patrol speed ratios and for given values of the average detection probability (which is obtained by averaging the detection probabilities over the target horizontal distances from the center of patrol path), the actual patrol length is calculated which maximizes the effective patrol length in the "worst case" of the back-and-forth pattern. Using these maximized values of the effective patrol length, we discuss the effects of varying parameters, such as target-to-patrol speed ratio, target heading, radar detection range.

To illustrate possible use of the derived equations and graphs, numerical results are given for two hypothetical examples: a patrol system in continental air defense and a limited war air defense counter-insurgence type of operation. However the ideas have more general applicability than military defense; they can be useful for search problems in operations research or analysis involving a searcher, moving in a repetitious pattern over a segment of boundary, and one or more moving object being searched for which is expected to cross that boundary segment.

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(All distances are given in units of R, the radar detection range)

A — full-cycle target region

D — distance that target moves in a half-cycle period

D_o — length representing totality of all initial target points at distance s (in A) detected by the patrol initially at left end of patrol path

D_o' — length representing totality of all initial target points at distance s (in A) detected by the patrol initially at arbitrary point of its cycl

l — length of patrol path (actual distance moved by patrol)

l_o — length representing totality of all initial patrol points which detect target initially at the point $T(x, y)$

L — effective patrol length (distance policed by patrol)

L MAXIMUM

maximum value of L for a given value of γ

n — number of patrol aircraft

N — minimum number of patrol aircraft required (a function of $\bar{p}$, v_t/v L MAXIMUM)

p — probability of detection (chance that a target initially at point T wi be detected by the patrol)

P_1 — proportion of initial patrol positions in a cycle which detect a targ initially at the point $T(x, y)$

P_2 — proportion of initial target positions, at distance s, detected by th patrol initially at left end of patrol path

P_2' — proportion of initial target positions, at distance s, detected by th patrol initially at arbitrary point of its cycle

$\bar{P}$ — average probability of detection (proportion of A that will be detec by the patrol initially at point P)

$p(s, l, \rho)$ — probability of detection for back-and-forth pattern and $\psi_t = \dfrac{3\pi}{2}$ (wor case)

List of Symbols (continued)

$\bar{P}(L, \ell, \rho)$ average probability of detection for back-and-forth pattern and $\psi_t = \dfrac{3\pi}{2}$ (worst case)

P patrol position

$P(x_p, 0)$ coordinates of initial position of patrol (back-and-forth pattern)

$P_0(-\dfrac{\ell}{2}, 0)$ coordinates of left end of patrol path (back-and-forth pattern)

$P_1(\dfrac{\ell}{2}, 0)$ coordinates of right end of patrol path (back-and-forth pattern)

R radar detection range (radius of detection circle)

s target horizontal distance from center of patrol path

T target position

$T(x, y)$ coordinates of initial position of target

v_p patrol speed

v_t target speed

ρ relative velocity heading for $\psi_p = 0$, $\psi_t = \dfrac{3\pi}{2}$ (worst case)

ρ_1 relative velocity heading for $\psi_p = 0$, ψ_t arbitrary

ρ_2 relative velocity heading for $\psi_p = \pi$, ψ_t arbitrary

ψ_p patrol heading

ψ_t target heading

ψ_r relative velocity heading of patrol with respect to target

1. STATEMENT OF PROBLEM

If an aircraft is flying back and forth patrolling a limited segment of a
boundary, how well can one expect to detect an approaching aerial target?
The practical problem is to optimize the detection capability of the patrol.
The theoretical problem is to find a suitable mathematical model which
will lead to meaningful interpretations of the real-life situation. In this
paper a basic detection model is formulated for repetitious patrol patterns.
Upon this basis is developed a measure of detection capability of the patrol--
the probability that a target at a given initial position is detected. To suit
the needs of the practical problem, this detection probability is maximized.
The concepts and numerical results can be applied to a wide range of
detection problems involving repetitious patrol patterns. Some examples
of repetitious patrol patterns are shown in Figure 1.

back-and-forth

rectangular

cross-over
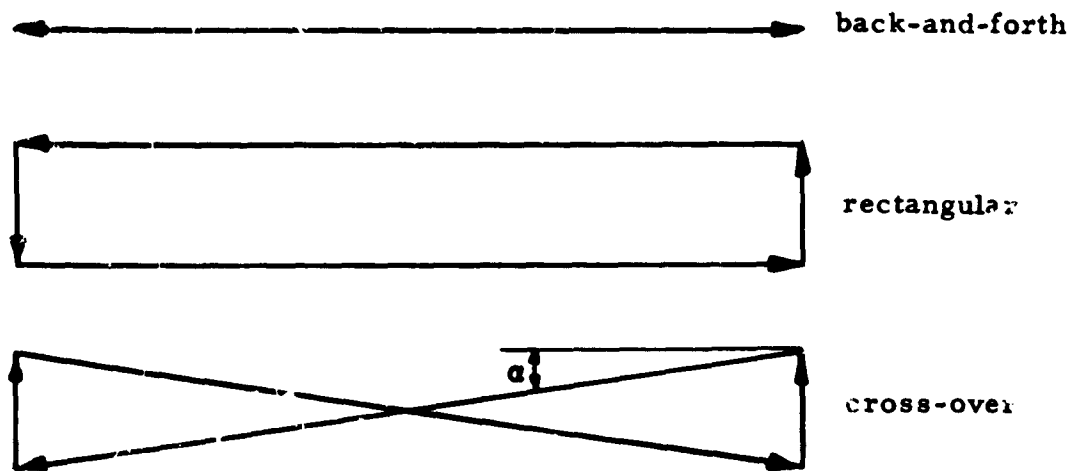
FIGURE 1. PATROL PATTERNS

It is well known that an aerial detection system which is located on

the ground is sadly deficient in detection of approaching low-altitude aerial

targets because its detection range is limited by the horizon due to the

curved surface of the earth; the air patrol, therefore, is a very appealing

idea since the earth no longer presents such an obstacle to a radar beam

emanating from some high altitude. Figure 2 shows the maximum line-of-

sight detection range vs (low) target altitude for a ground radar and for an

airborne radar at 40,000 feet altitude. Sometimes zealous advocates of

air patrols overestimate the detection capability of such patrols. For ex-

ample, claims have been made to the effect that aerial patrols with circular

radar can detect all targets up to supersonic speeds. The results of this

paper will show that such a statement requires qualification, that for target

speeds greater than or near the patrol speed there may be limitations in

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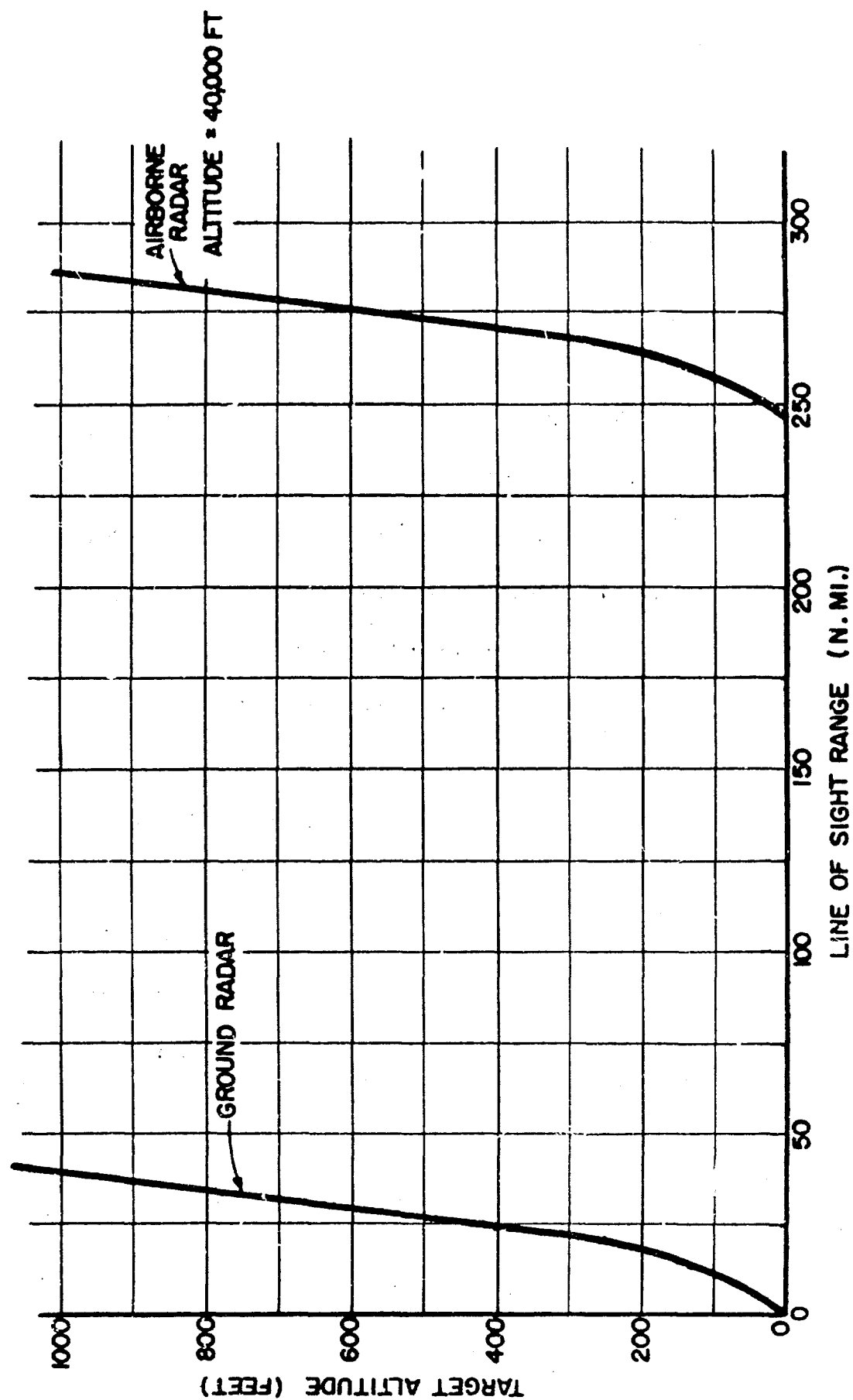
FIGURE 2. MAXIMUM LINE-OF-SIGHT RANGE vs. TARGET ALTITUDE

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air patrol detection capability. Nevertheless, air patrols do represent an advantage over ground radar and their potential should be explored in a careful scientific manner.

While undoubtedly many technical reports of the "crash program" type purport to provide solutions to this type of detection problem, to the authors' knowledge only one such attempt of a serious nature is described in the research literature, that of Koopman [1] and his Operations Research Group [3]. That portion of the ORG's work which has direct bearing on theory for repetitious patrol patterns is given on pages 106-7 under the subheading of "Barrier Patrols" in Chapter 7 of [3]. The effectiveness of a back-and-forth pattern was compared with that of a cross-over pattern (see Figure 1) for constant patrol and target speeds. However, that commendable work was based primarily on detection of slow-moving ships, and can not be applied to detection of targets moving faster than the patrol. Furthermore, the angle of the cross-over patrol pattern was defined to be the inverse sine of the ratio of target-to-patrol speeds, a choice which was unnecessarily restrictive from the theoretical viewpoint. It will be seen that the methods to be described in this paper are not hampered by these restrictions.

The following assumptions are made in this paper. The patrol and target speeds are constant. The patrol plane is flying continuously. The radar detection capabilities are independent of differences of altitude between target and patrol, so that a two-dimensional representation of detection is

adequate. The radar detection pattern is circular with radius R; R is also known as the "radar detection range". If T, the point corresponding to a target position, lies on or inside the circle of detection, the target is detected; if T lies outside the circle of detection then the target is not detected.

We also assume, for convenience of calculation, that a target is equally likely to be located at any point of the region of interest. Though this uniform or rectangular probability distribution has been assumed, theoretically one can take into account any general probability distribution of targets, by weighting the detection probability by the probability distribution of the targets. We discuss only the back-and-forth pattern since the discussion becomes rather involved. Also when developing detection probability (Section 3 and 4) we consider only the "worst" case with respect to detection by the patrol; namely the one in which the target heading is perpendicular to the patrol path. The basic concepts (Section 2) do not really depend on the particular target and patrol velocities, nor on the particular shape of radar detection pattern, nor on the particular patrol path pattern as long as it is repeatable, nor on the particular target heading. These concepts are therefore useful for more general situations, including cross-over patrol patterns of a more general type than that discussed by Koopman's ORG Group.

The primary reason for the above restrictions is mathematical simplicity. The problem becomes one involving constancy and symmetry; the geometry, especially, can be used to good advantage. However, there are other cogent

reasons for such a choice. circular radar detection pattern is already being developed for operational use in a Navy plane; this system was described in an issue of Aviation Week. It would be nice to be able to apply numerical results of the developed theory directly to an analysis of the capabilities of such a system. From operational considerations, a back-and-forth patrol path, being the simplest and easiest to describe, will be most reliably executed under combat conditions and stress. The worst case is also a very important one. It tells the strategist, or planner, or evaluator, what is the lower bound on detection one can expect of patrol performance. Of course, for targets approaching with headings other than perpendicular, the patrol may do much better. However, from the target's point of view, the target wants to take the least time to cross the patrol path, to avoid detection, and will tend to approach at an angle which is as close to perpendicular as possible. Therefore, results obtained for the worst case will not be far from those in actual combat conditions.

We take as a premise that the two most important measures of detection capability are: probability of detection (suitably defined) and duration of detection (detection time). Of the two measures, the latter is more troublesome to analyze. In this paper we shall be primarily concerned with a study of the first measure, although in our discussion of basic concepts we shall touch upon some general ideas concerning detection time also. We define "probability of detection" in this paper as the probability that a target initially

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at some given position is detected. This detection probability expresses numerically the chances that the patrol will detect a target position. The probabilities of detection we obtain for the worst case furnish a lower bound for probabilities of detection corresponding to target heading other than perpendicular.

The parameters to be studied are listed as follows with their symbols: length of patrol path, ℓ ; radar detection range, R; patrol speed, v_p; target speed, v_t; relative velocity heading of patrol to target, ρ . We shall introduce a new parameter, the effective patrol length, L, and also consider the horizontal distance from the center of the patrol path, s. In the computation of the probability of detection in Section 4, all distance measures are put in units of R.

The target is assumed to be initially outside of the region whose boundary is being patrolled. We distinguish between two situations:

Case 1: target initially at distance from the patrol path greater than R cos ρ (long-range targets).

Case II: target initially at distance from the patrol path less than R cos ρ (short-range targets).

For long-range targets, Case I, the detection probability is independent of distance from patrol path. For short-range targets, Case II, the detection probability depends on distance from patrol path. Case II can also be adequately handled by our theory; the relative detection region will not be exactly the same as for Case I, and consideration of Case II will lead to

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lower probabilities of detection than for Case I. The short-range case occurs in a practical situation, for example, when an enemy airfield is located at a distance from the patrol path which is less than $R \cos \rho$. On some occasions, especially in limited war, airplanes may take off from such fields at any time in an attempt to cross over the boundary being patrolled. However, space in the paper being at a premium, and our main objective being the clarification of basic concepts and how they are used to obtain a useful measure of detection capability, we will consider only Case I in this paper.

2. BASIC CONCEPTS.

A back-and-forth patrol, whose initial position is represented by a point P, moves along a straight-line path between two points P_0, P_1 at a constant velocity $\vec{v}_p$ (heading ψ_p) as shown in Figure 3.
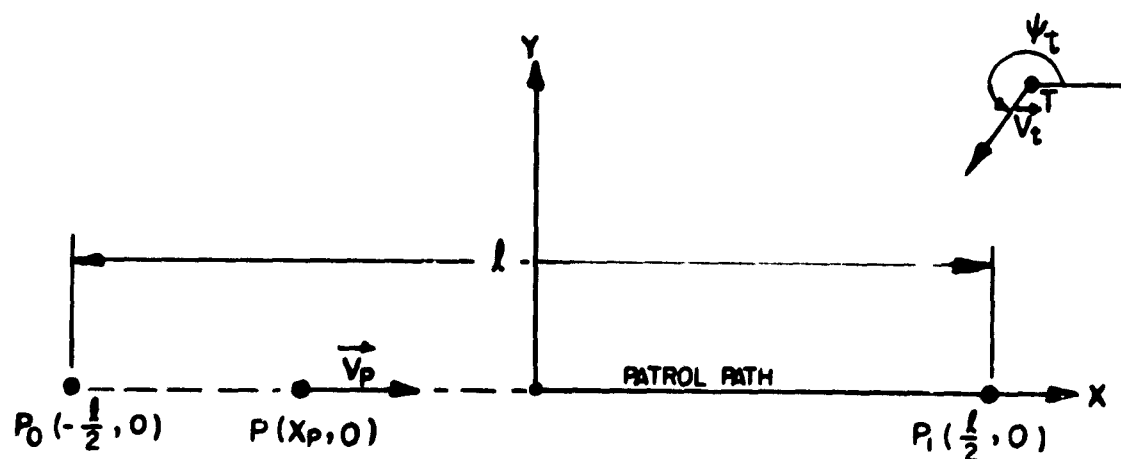
FIGURE 3. COORDINATE SYSTEM

The points are at a distance ℓ apart. We choose a fixed rectangular coordinate system (x, y) with positive x-axis along the horizontal, and positive y-axis in the up-direction as indicated in Figure 3. By convention, all heading angles measured counterclockwise from the x-axis are considered positive. With no real loss of generality, the patrol path is assumed to be along the horizontal; so that when the patrol moves to the right from P_0 to P_1, ψ_p is taken equal to zero; when the patrol moves to the left from P_1 to P_0, ψ_p is taken equal to π. Also without loss of generality, the origin is chosen at 0, the midpoint of $\overline{P_0 P_1}$. Then P_0 is given by $(-\ell/2, 0)$, and P_1 by $(\ell/2, 0)$. The patrol initial position is given by $P(x_p, 0)$.

The target, whose initial position is represented by a point $T, T(x, y)$, moves downward in the general negative y-direction with a constant velocity $\vec{v}_t$ (heading ψ_t); ψ_t is in the range $\pi \leq \psi_t \leq 2\pi$. The relative velocity $\vec{v}_r$ of P with respect to T (heading ψ_r) is given by

$$\vec{v}_r = \vec{v}_p - \vec{v}_t \quad ;$$

it is indicated in Figure 4 for $\psi_p = 0$ and in Figure 5 for $\psi_p = \pi$.
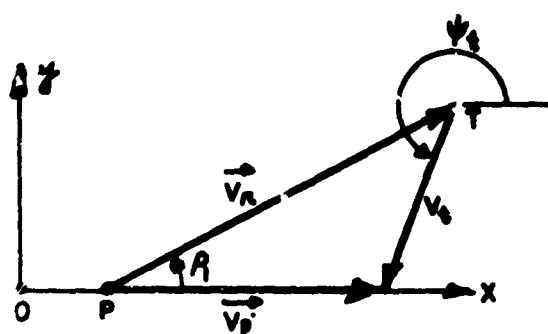
FIGURE 4. VECTOR DIAGRAM
OF VELOCITIES FOR $\psi_p = 0$
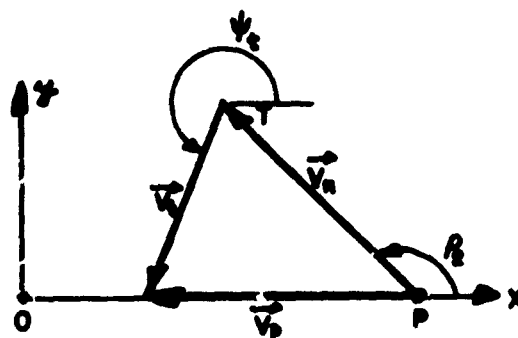
FIGURE 5. VECTOR DIAGRAM
OF VELOCITIES FOR $\psi_p = \pi$

The angles ρ_1, ρ_2 are the specific values of ψ_r when $\psi_p = 0$, $\psi_p = \pi$, respectively. These angles are also indicated in Figures 4 and 5.

If $\psi_t = \frac{3\pi}{2}$ (target moving perpendicular to the patrol path, i.e. target moving vertically down), then $\rho_1 = \pi - \rho_2 = \rho$. Geometrically, this is the "symmetric" case. It is also what we referred to as the "worst case" for the patrol.

By the cosine law,

$$v_r = [v_t^2 + v_p^2 - 2v_t v_p \cos(\psi_p - \psi_t)]^{\frac{1}{2}} \quad (2.1)$$

By the sine law,

$$\frac{v_p}{\sin[\psi_t - \psi_r - \pi]} = \frac{v_t}{\sin[\psi_r - \psi_p]} \quad , \quad \text{yielding}$$

$$\psi_r = \tan^{-1} \frac{v_t \sin \psi_t - v_p \sin \psi_p}{v_t \cos \psi_t - v_p \cos \psi_p} \quad (2.2)$$

For the worst case ($\psi_t = \frac{3\pi}{2}$) it is easily seen that

$$\tan \rho = \frac{v_t}{v_p} \qquad \text{or} \qquad \rho = \tan^{-1} \frac{v_t}{v_p} \quad (2.3)$$

BASIC CONCEPT NUMBER I: EFFECTIVE PATROL LENGTH, L .

Because of the radar detection range, the patrol has a detection capability beyond both ends of its path. In fact, it can sometimes detect a target at a horizontal distance of R beyond either end. One is thus led to

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consider a length which takes into account the fact that the patrol can detect beyond the ends. Denote this length by L. It is defined as the length which the patrol polices. In theory, L may also be shorter than the length which the patrol flies. A "desirable" value of L is one which is greater than ℓ , of course. We call L the "effective patrol length".

BASIC CONCEPT NUMBER II: FULL-CYCLE TARGET REGION. A.

The full-cycle period is the time it takes the patrol to move from point P(heading ψ_p) through a full cycle of positions till it returns to the same point P and me heading. For the back-and-forth patrol pattern this full cycle of positions of the patrol is of length 2 ℓ. Let D be the distance that the target moves during a half-cycle period. Then

$$D = \ell \frac{v_t}{v_p} \tag{2.4}$$

Corresponding to the full-cycle patrol "length" of 2 ℓ , the target positions range over a full cycle of positions of length 2D.

The lines through (-L/2, 0) and through(L/2, 0), making angles ($\psi_t - \pi$) with the positive x-axis, form the left and right target boundaries of the area which the patrol will police. Any target within these boundaries comes within the jurisdiction of the given patrol. The region enclosed by these two boundaries and the horizontal lines

$$y = R \left| \cos \psi_r \right| \; , \; y = R \left| \cos \psi_r \right| + 2D \sin (\psi_t - \pi)$$

is called the "full-cycle target region, A" and is shown by the shaded region in Figure 6.

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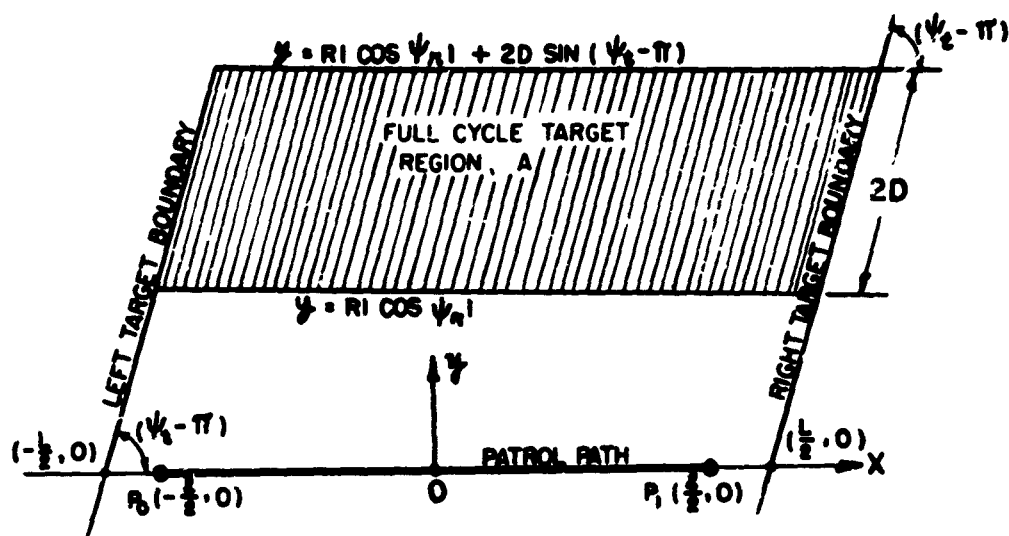
FIGURE 6. FULL-CYCLE TARGET REGION

Since any target can be detected _only_ when it is within detection range, it is evident that _only_ the full-cycle area of targets, A, need be investigated for various initial positions of the patrol within the full cycle, ranging from $P_0(-\ell/2, 0)$ to $P_1(\ell/2, 0)$ and back to $P_0(-\ell/2, 0)$ over a total (full cycle) length of 2ℓ. The reasoning is analogous to that for a sine curve where all the properties are given in one full-cycle period.

BASIC CONCEPT NUMBER III: DUALITY RELATIONSHIP IN RELATIVE MOTION.

Besides the obvious (but not necessarily the easiest) way of letting both target and patrol move at the same time while studying their kinematics, we can look alternatively at relative motion in two dualistic ways. The first approach is to keep the target position (or point) fixed, with only the patrol

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and its detection circle moving relatively to the target. The second approach is to keep the patrol fixed, with only the target moving ralatively to the patrol. The latter simulates what could be seen on the patrol's radar scope and is more convenient for developing properties about detection time. On the other hand, the first is much simpler and more direct for developing probability of detection; we accordingly pursue the first approach. Nevertheless, the important point to keep in mind is that both approaches are equally valid and that sometimes one of the approaches is more convenient to use than the other, depending on what information we want. These dual aspects are basic to our theory for detection capability of a repetitious patrol. Even in using the first approach (target fixed, patrol moving relatively) we discuss it in a dualistic manner. We can talk about proportion of initial patrol points (when varied over the cycle) for which a given initial target point in A will be detected: or we can talk about proportion of initial target points along a line in A (parallel to the target boundaries) which a given initial patrol point will detect. The two proportions (probabilities) are identical, and the two notions are duals of one another.

BASIC CONCEPT NUMBER IV. DETECTION REGION.

At the time the target is initially at point $T(x, y)$, the patrol is initially at point $P(x_p, 0)$ as shown in Figures 7 and 8. The detection region is defined as follows: any target point initially on the boundary or inside the
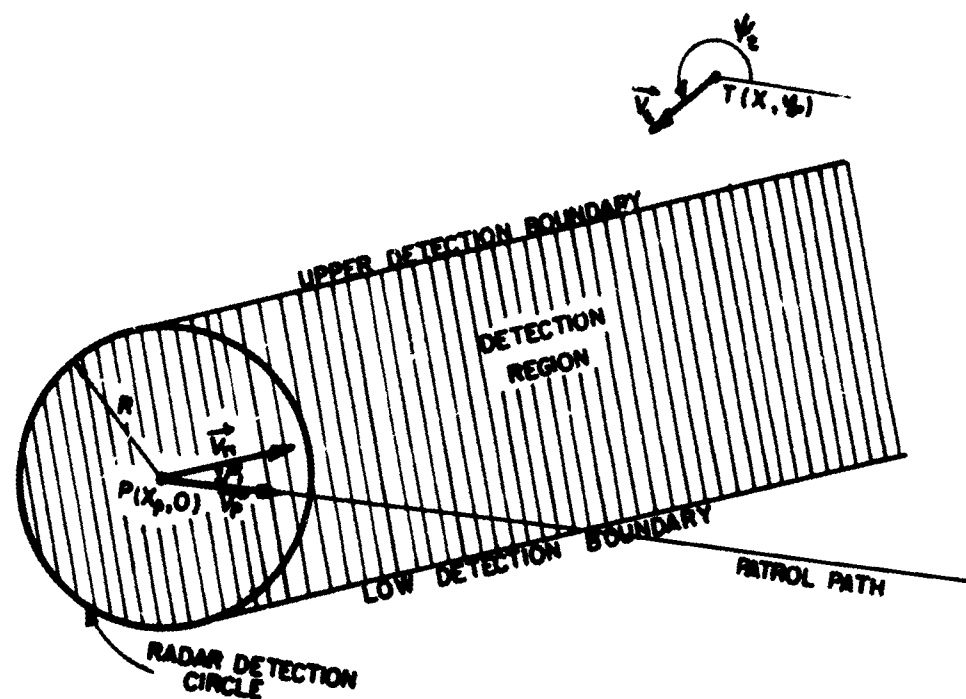
FIGURE 7. DETECTION REGION FOR $\psi_p = 0$
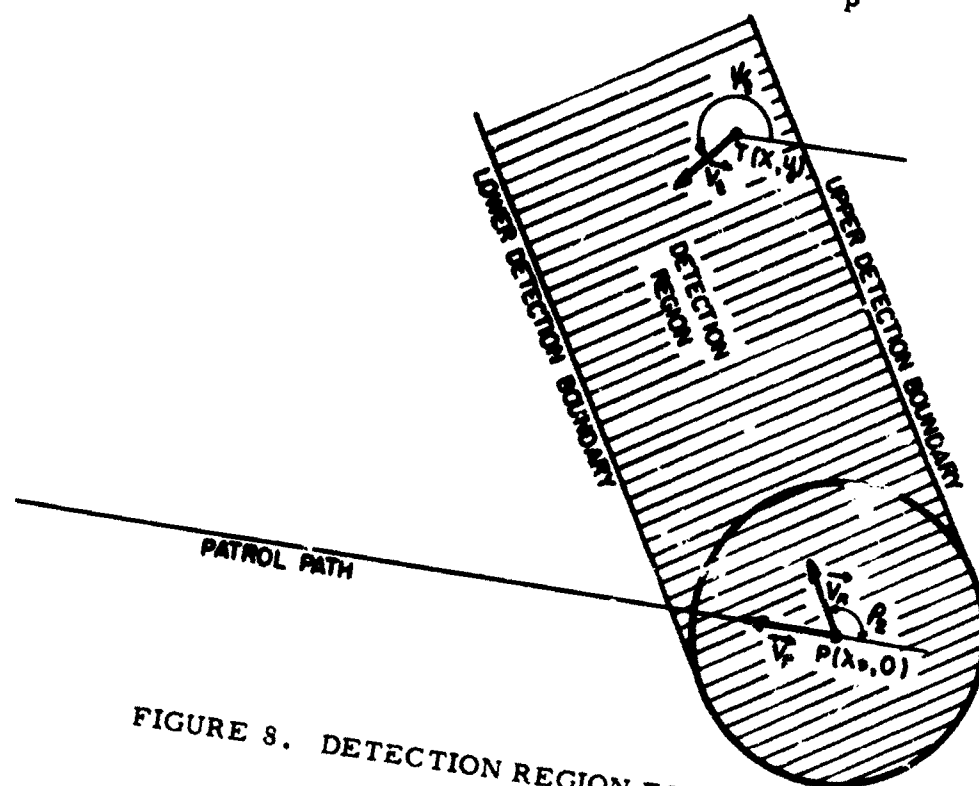
FIGURE 8. DETECTION REGION FOR $\psi_p = \pi$

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region at the time the patrol is initially at point P will eventually be detected; any target point initially outside of the region at the time the patrol is initially at point P will not be detected.* The detection region for the patrol initially at point P, for $\psi_p = 0$, is shown in Figure 7. The detection region for the patrol initially at point P, for $\psi_p = \pi$, is shown in Figure 8. A derivation of the straight line boundaries of the detection region, for patrol initially at P, is given in the Appendix. It is shown in the Appendix that the slope of the straight line boundaries is $\tan \psi_r$, which is $\tan \rho_1$ for $\psi_p = 0$ as in Figure 7, and $\tan \rho_2$ for $\psi_p = \pi$ as in Figure 8. For the back-and-forth pattern, and for the worst case ($\psi_t = \frac{3\pi}{2}$), the shaded portion of Figure 9 shows the detection region for the patrol initially at P_0 for one cycle; the shaded portion of Figure 10 shows the detection region for the patrol initially at arbitrary P for one cycle. Note that the detection region, which depends on the initial position P of the patrol, does not directly depend on the target heading ψ_t; the dependence is really on the relative velocity heading ψ_r. In Figures 9 and 10 the shaded portions of the full-cycle target region A will be detected, the unshaded portions of the full-cycle target region A will not be detected. In other words, those initial target points in A which lie inside the detection region (for the patrol initially at point P) will be detected, those initial target points in A which lie outside the

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* For any initial target point that will be detected, i. e. lying on the boundary or inside the region, the number of times it is detected and the corresponding durations can also be exactly determined, though the procedure becomes computationally involved.

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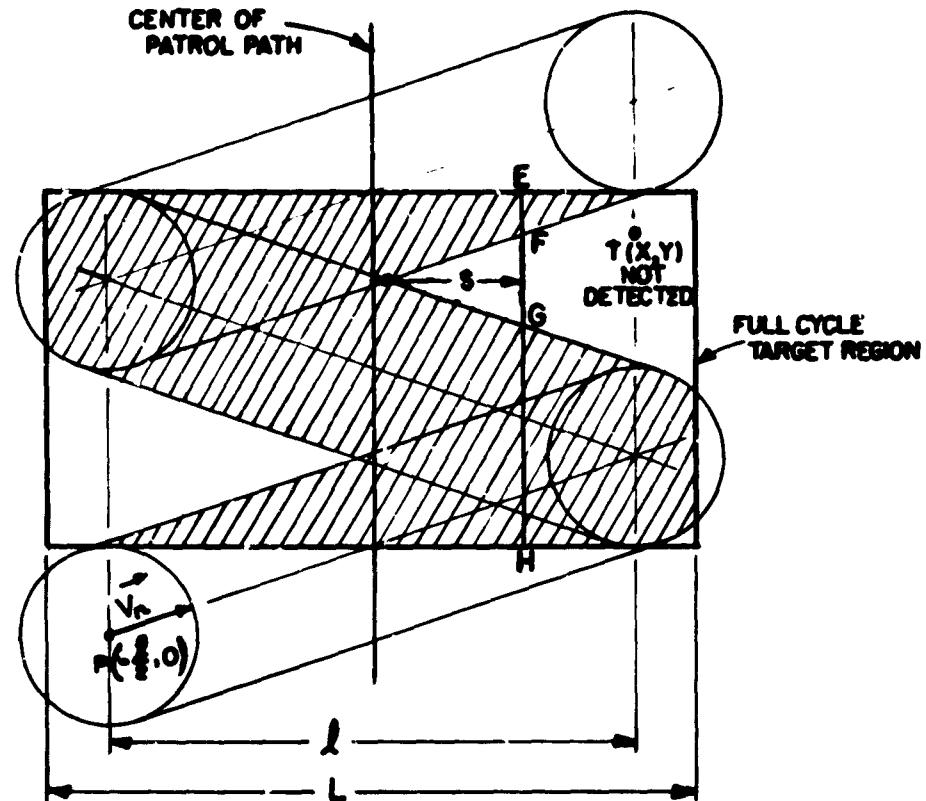
FIGURE 9. DETECTION REGION SUPER IMPOSED ON FULL CYCLE
TARGET REGION FOR PATROL INITIALLY AT P_0
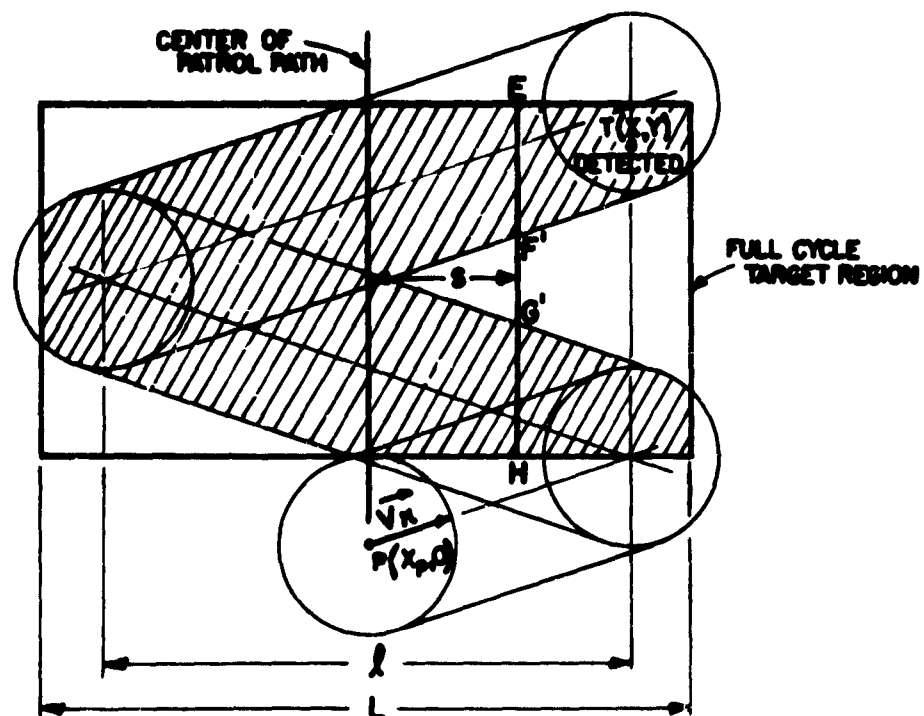
FIGURE 10. DETECTION REGION SUPER IMPOSED ON FULL CYCLE
TARGET REGION FOR PATROL INITIALLY AT ARBITRARY POINT P

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detection region (for the patrol initially at point P) will not be detected.

Figure 9 is the detection region, for the patrol initially at P_0 $(-\ell/2, 0)$, superimposed on the full-cycle target area A. Thus, Figure 9 is the detection picture corresponding to the target at any initial point in A for the patrol starting at point P_0. Similarly, Figure 10 is the detection region, for the patrol initially at $P(x_p, 0)$, superimposed on the target area A. Thus, Figure 10 is the detection picture corresponding to the target at any initial point in A for the patrol starting at arbitrary point P.

We have now described the necessary theoretical ideas basic to evaluating patrol detection capability for repetitious patrol patterns.

3. DETECTION PROBABILITY FOR THE WORST CASE.

The discussion in this section and the following section (Section 4) will deal only with the worst (symmetric) case, $\psi_t = \dfrac{3\pi}{2}$.

We have developed a simple mechanism for determining whether the patrol starting at point P will or will not detect the target starting at point T. However, the important consideration in making statements about probability is NOT whether a particular patrol position will or will not detect a particular target position. In reality it is usually not known at what particular position the patrol will be when a target starts from some point; even if this is known, it may not be at all desirable operationally to control the patrol's movements except with regard to rather general instructions. To be really effective, the patrol must be capable of detecting

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any number of targets crossing the boundary at any time. Also, it is

often not exactly known what particular locations the targets may emanate

from. Therefore, it is of utmost value to find a probability measure which

is <u>invariant</u> with respect to patrol position (and, if possible, also invariant

with respect to target position). In accordance with these notions we

proceed to derive a formula for probability of detection, remembering that

it has been defined on page 6 as the "<u>probability that a target initially at</u>

<u>some given position is detected</u>".

Let s be the horizontal distance of the initial target point $T(x, y)$ from

the center of the patrol path. Then s is the absolute value of x. For the

moment, let us fix s.

Consider the fixed initial target point $T(x, y)$. For the back-and-forth

pattern, the cycle of possible initial patrol positions (points) is of length

2ℓ, where ℓ is the actual patrol length. Let ℓ_0 be the length representing

the totality of all initial patrol points which detect a target initially at the

fixed point $T(x, y)$. Let p_1 be the proportion of initial patrol positions in a

cycle which detect a target initially at the fixed point $T(x, y)$. Then p_1 is

given by

$$p_1 = \frac{\ell_0}{2\ell} \tag{3.1}$$

<u>The required formula for the "probability of detection" as defined on page 6</u>

<u>is given by (3.1).</u>

Likewise, for the same fixed value of s, consider the dual situation of

a fixed initial patrol point at $P_0(-\ell/2, 0)$ as shown in Figure 9. Consider all targets at the same fixed horizontal distance s from the center of the patrol path. For the back-and-forth pattern, the cycle of possible initial target positions (points) is of length $2D$. It is the line-segment $\overline{EH}$ in Figure 9. Let D_0 be the totality of all initial target points at distance s detected by the patrol starting at fixed point P_0. Then D_0 is given by

$$D_0 = \overline{EF} + \overline{GH} \tag{3.2}$$

in Figure 9. Let p_2 be the proportion of initial target points (distance s) detected by the patrol initially at fixed point P_0. Then p_2 is given by

$$P_2 = \frac{D_0}{2D} \tag{3.3}$$

Similarly, consider another fixed initial patrol point $P(x_p, 0)$ as in Figure 10. Then according to (3.3), p_2' is the ratio $\frac{D_0'}{2D}$, where D_0' is given by

$$D_0' = \overline{EF'} + \overline{G'H} \tag{3.4}$$

in Figure 10. Since $\overline{FG} = \overline{F'G'}$, it follows that D_0 in (3.2) is the SAME length as D_0' in (3.4). Hence, we have shown that p_2 is identically equal to p_2', i.e., that p_2 is _independent_ of the initial patrol position P.

But $\ell_0 = D_0 \tan \rho$, and $\ell = D \tan \rho$. It follows that p_1 in (3.1) is identical with p_2 in (3.3), i.e., $p_1 = p_2 = p$.

Therefore, the probability of detection, p, has two equivalent inter-pretations. It is the proportion of (initial) patrol points which detect any arbitrary (initial) target point at horizontal distance s from center of the patrol path. Alternatively, it is the proportion of (initial) target points at horizontal distance s from center of the patrol path detecteᴜ ᵞy the patrol at arbitrary (initial) point P. Therefore, the probability of detection, p, is invariant with respect to (initial) patrol position, and is a function only of the following three parameters: s, the horizontal distance of the (initial) target point from the center of the patrol patn; l, the actual patrol length; ρ, the inverse tangent of the ratio of target-to-patrol speeds. We accord-ingly replace p by $p(s, l, \rho)$.

Figures 11, 12, 13, indicate, for $v_t/v_p = 1/3$, the dependence of the probability of detection on l and ρ according as l lies in three ranges:

$$0 \leq l \leq \frac{\cos \rho}{\tan \rho} \text{ (Case A)}, \quad \frac{\cos \rho}{\tan \rho} \leq l \leq \frac{2}{\sin \rho} \text{ (Case B)}, \quad \frac{2}{\sin \rho} \leq l \text{ (Case C)}.$$

Figures 11, 12, 13 also indicates, for $v_t/v_p = 1/3$, the dependence of the prob-ability of detection on s according as s lies in eight subranges denoted by A_1, A_3^* in Figure 11 corresponding to Case A; denoted by B_1, B_2, B_3 in Figure 12 corresponding to Case B; denoted by C_1, C_2, C_3 in Figure 13 corresponding to Case C.

For the worst case, $\psi_t = \frac{3\pi}{2}$, and since $\tan \rho = \frac{v_t}{v_p}$, the formula of

* This numbering system was adopted for easy cross-reference with Case B and Case C. There is NO "Case A2".

$$\frac{V_T}{V_P} = \frac{1}{3}$$

$$\psi = \frac{3\pi}{2}$$

R = RADAR DETECTION RANGE

ρ = RELATIVE VELOCITY HEADING

CENTER OF PATROL PATH

2D = FULL CYCLE DISTANCE

$\ell = 2R$ PATROL PATH

REGION OF NO DETECTION

A_1

A_3

FIGURE 11. DETECTION REGION SUPER IMPOSED ON FULL CYCLE TARGET REGION FOR CASE A: $0 \leq \ell \leq \dfrac{\cos\rho}{\tan\rho}$
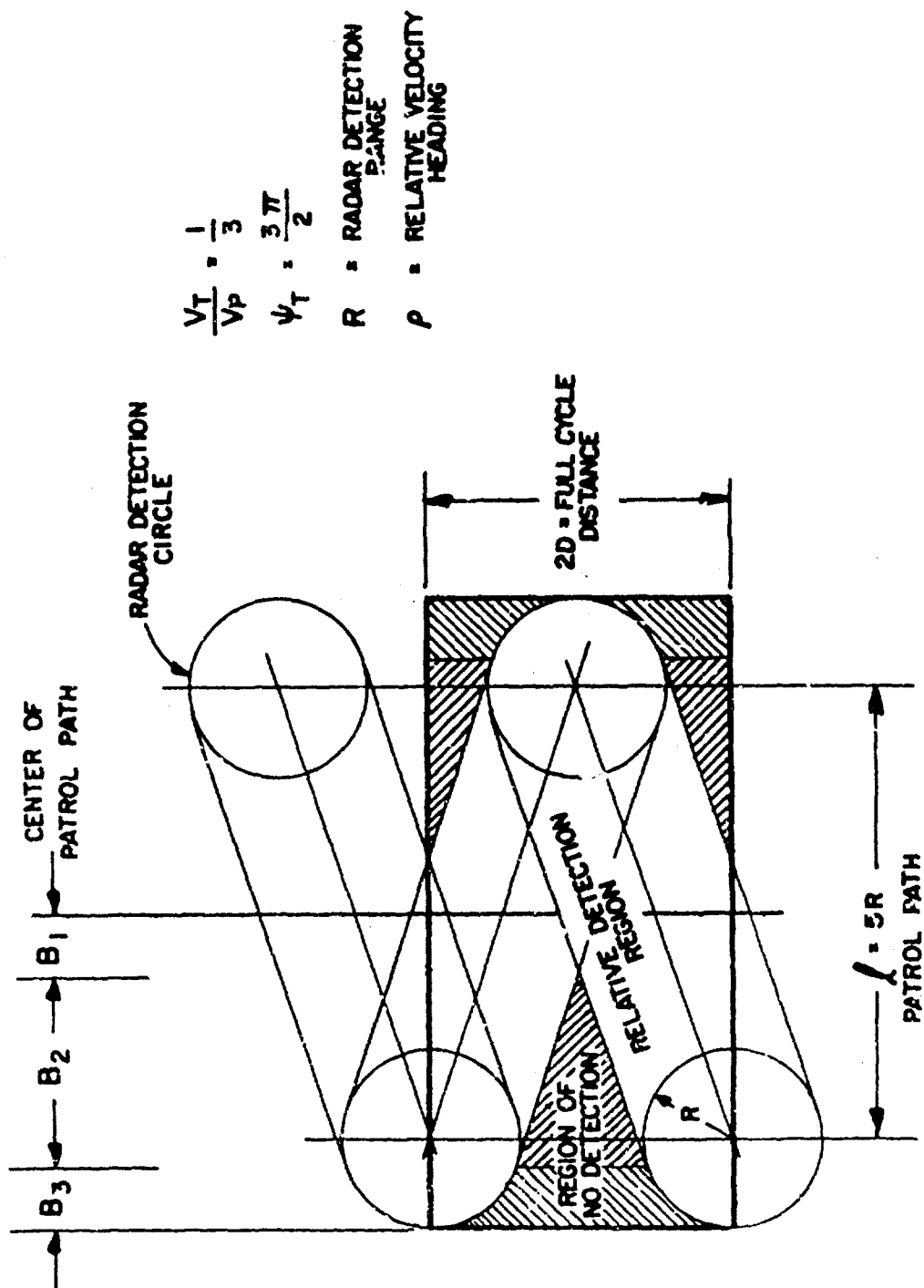
FIGURE 12. DETECTION REGION SUPERIMPOSED ON FULL
CYCLE TARGET REGION FOR CASE B: $\dfrac{\cos\rho}{\tan\rho} \leq \ell \leq \dfrac{2}{\sin\rho}$
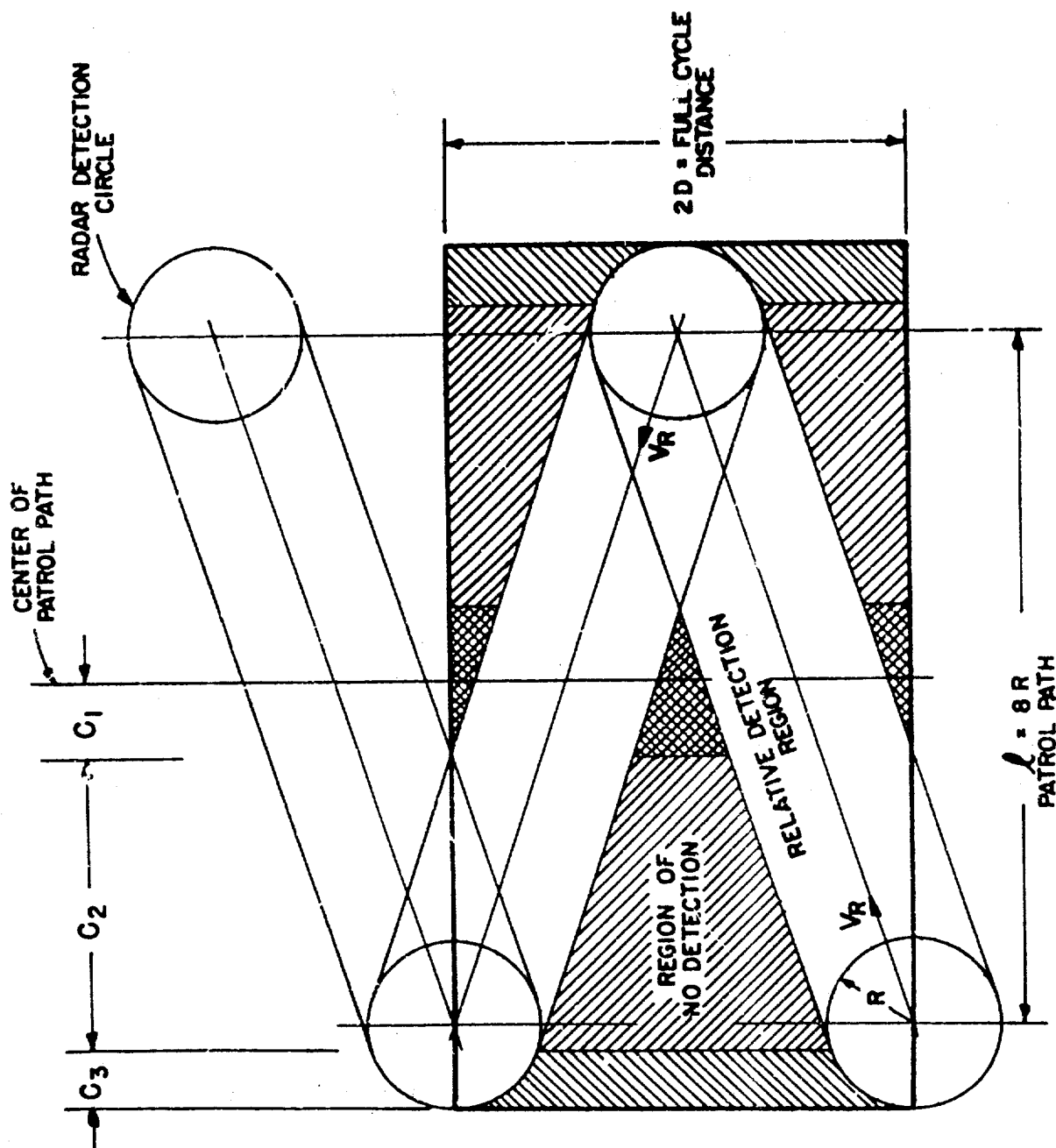
$\dfrac{V_T}{V_P} = \dfrac{1}{3}$

$\psi_T = \dfrac{3\pi}{2}$

R = RADAR DETECTION RANGE

ρ = RELATIVE VELOCITY HEADING

CENTER OF PATROL PATH

B_3 B_2 B_1

RADAR DETECTION CIRCLE

2D = FULL CYCLE DISTANCE

RELATIVE DETECTION REGION

REGION OF NO DETECTION

$\ell = 5R$
PATROL PATH

R

FIGURE 13. DETECTION REGION SUPERIMPOSED ON FULL CYCLE TARGET REGION FOR CASE C: $\ell \geq \dfrac{2}{\sin \rho}$

the detection probability p, for varying s, is obtained by applying (3.3) to

Figures 11, 12, 13   The formula for probability of detection $p(s, \ell, \rho)$ is

summarized below.   ALL DISTANCES ARE GIVEN IN UNITS OF R, THE

RADAR DETECTION RANGE.

CASE A: $0 \leq \ell \leq \dfrac{\cos \rho}{\tan \rho}$

CASE A1: $p(s, \ell, \rho) = 1$

$$\text{for } 0 \leq s \leq \frac{\ell}{2} + \sqrt{1 - (\ell \frac{v_t}{v_p})^2} \qquad (3.5a)$$

CASE A3*: $p(s, \ell, \rho) = \dfrac{1}{\ell \dfrac{v_t}{v_p}} \sqrt{1 - (s - \frac{\ell}{2})^2}$

$$\text{for } \frac{\ell}{2} + \sqrt{1 - (\ell \frac{v_t}{v_p})^2} \leq s \leq \frac{\ell}{2} + 1.$$

CASE B: $\dfrac{\cos \rho}{\tan \rho} \leq \ell \leq \dfrac{2}{\sin \rho}$

CASE B1: $p(s, \ell, \rho) = 1$

$$\text{for } 0 \leq s \leq \frac{1}{\sin \rho} - \frac{\ell}{2}$$

$$(3.5b)$$

CASE B2: $p(s, \ell, \rho) = 1 - \dfrac{(s + \frac{\ell}{2} - \frac{1}{\sin \rho})}{\ell}$

$$\text{for } \frac{1}{\sin \rho} - \frac{\ell}{2} \leq s \leq \frac{\ell}{2} + \sin \rho$$

---

* This numbering system was adopted for easy cross-reference wit   Case B
and Case C.   There is NO "Case A2".

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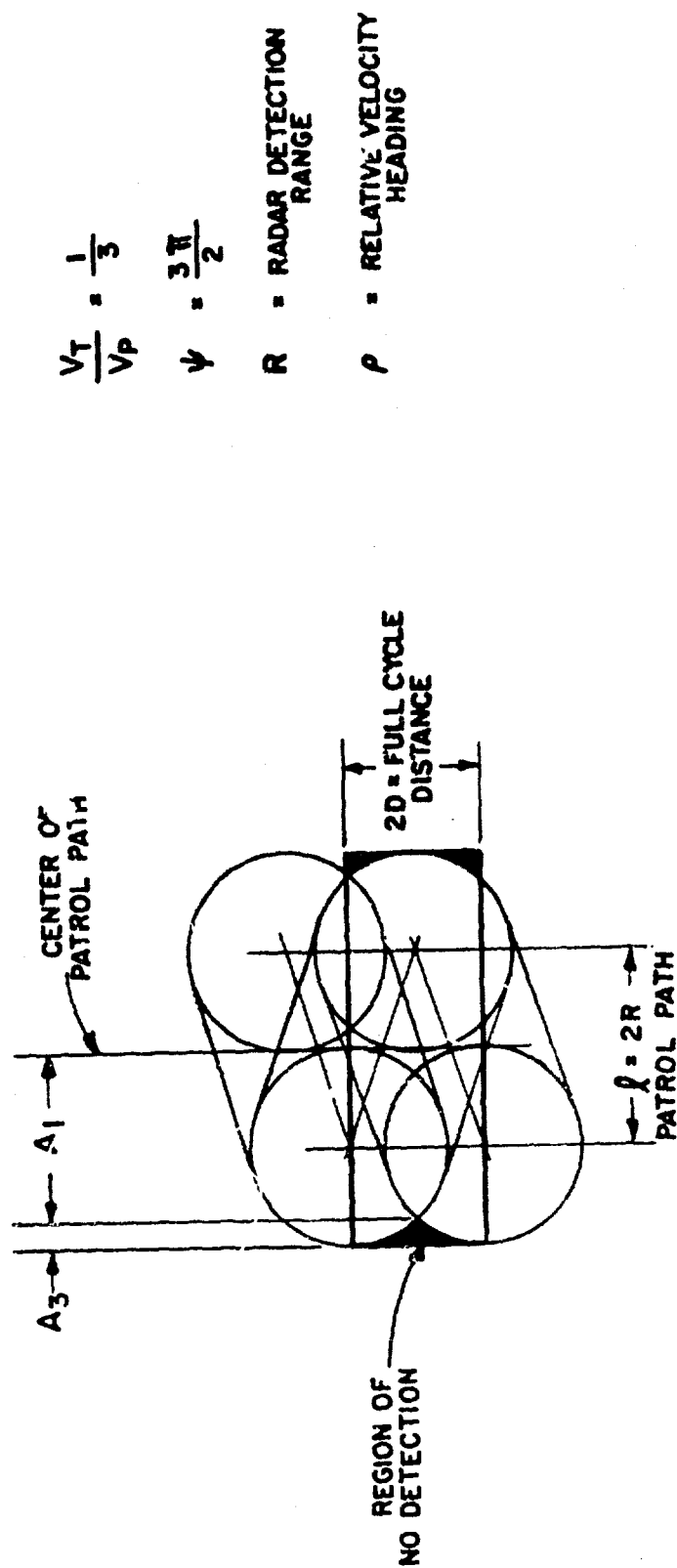CASE B3: $p(s, \ell, \rho) = \dfrac{1}{\ell\left(\dfrac{v_t}{v_p}\right)} \sqrt{1 - (s - \dfrac{\ell}{2})^2}$

for $\dfrac{\ell}{2} + \sin \rho \leq s \leq \dfrac{\ell}{2} + 1$

CASE C: $\ell \geq \dfrac{2}{\sin \rho}$

CASE C1: $p(s, \ell, \rho) = \dfrac{2}{\ell \sin \rho}$

for $0 \geq s \leq \dfrac{\ell}{2} - \dfrac{1}{\sin \rho}$

CASE C2: $p(s, \ell, \rho) = 1 - \dfrac{(s + \dfrac{\ell}{2} - \dfrac{1}{\sin \rho})}{\ell}$ (3.5c)

for $\dfrac{\ell}{2} - \dfrac{1}{\sin \rho} \leq s \leq \dfrac{\ell}{2} + \sin \rho$

CASE C3: $p(s, \ell, \rho) = \dfrac{1}{\ell \dfrac{v_t}{v_p}} \sqrt{1 - (s - \dfrac{\ell}{2})^2}$

for $\dfrac{\ell}{2} + \sin \rho \leq s \leq \dfrac{\ell}{2} + 1$

Figures 14, 15, 16 are curves (for various values of ℓ) of the probability of detection vs. normalized distance ($\dfrac{s \sin \rho}{R}$) from center of patrol path corresponding to $v_t/v_p = 1/3, 1, 3$, respectively. The particular values of ℓ shown were chosen so as to allow direct comparison among the three figures associated with the three speed ratios.
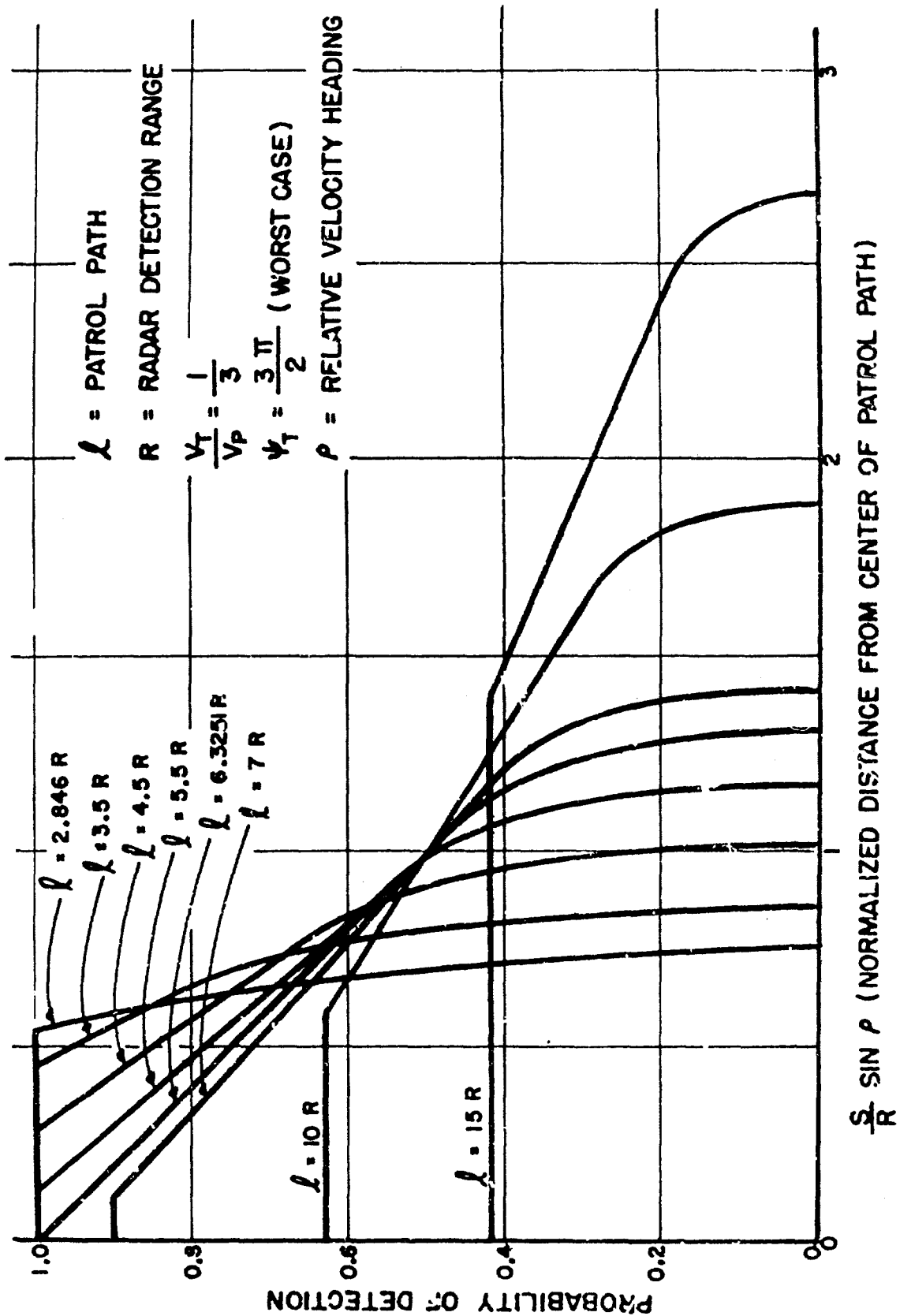
FIGURE 14. DETECTION PROBABILITY vs. DISTANCE FROM CENTER OF PATROL PATH FOR $v_t/v_p = 1/3$

FIGURE 15. DETECTION PROBABILITY vs. DISTANCE FROM
CENTER OF PATROL PATH FOR $v_t/v_p = 1$

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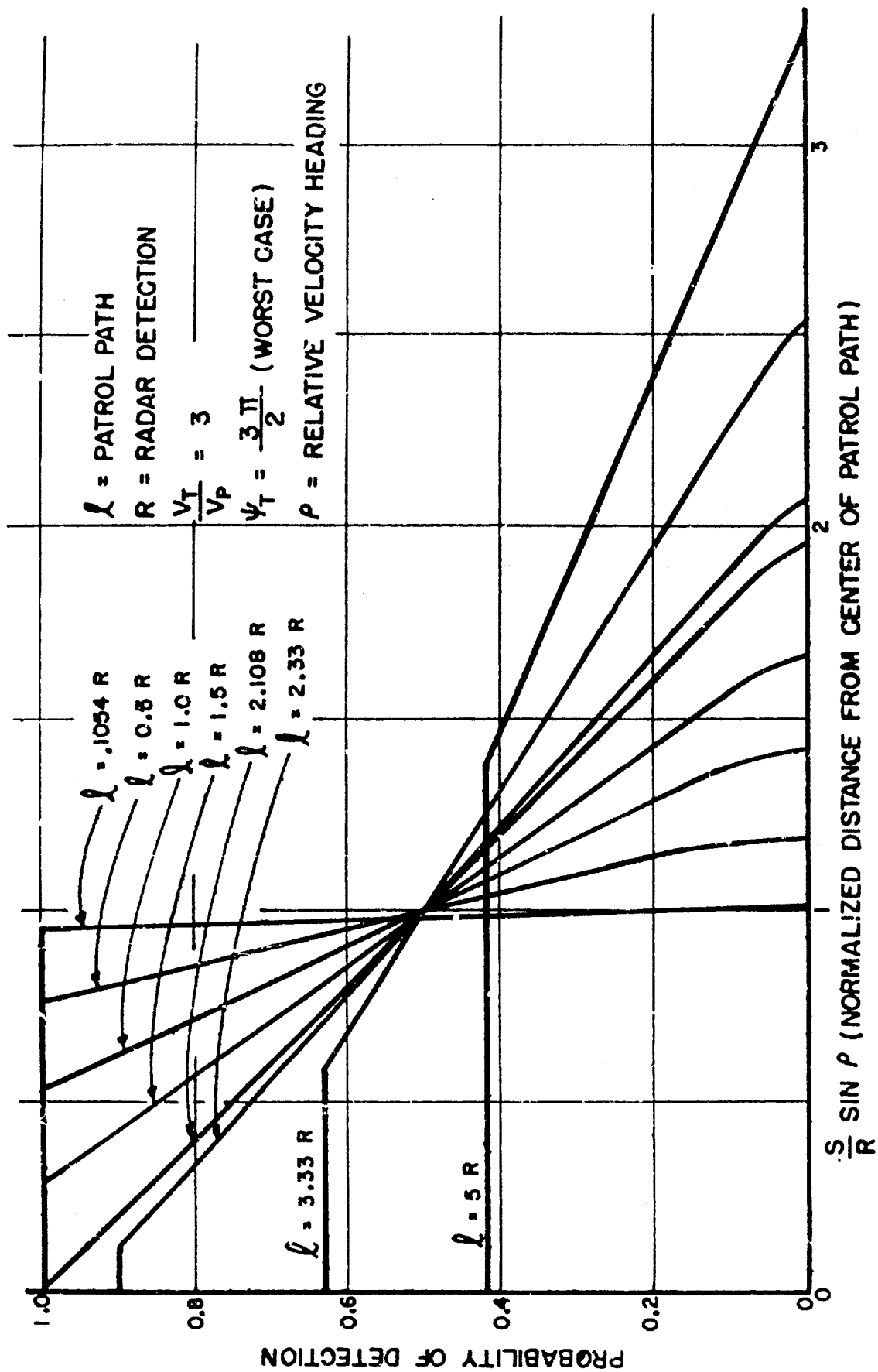
FIGURE 16. DETECTION PROBABILITY vs. DISTANCE FROM
CENTER OF PATROL PATH FOR $v_t/v_p = 3$

Legend within figure:

ℓ = PATROL PATH

R = RADAR DETECTION

$\dfrac{v_T}{v_P}$ = 3

$\psi_T = \dfrac{3\pi}{2}$ (WORST CASE)

ρ = RELATIVE VELOCITY HEADING

ℓ = .1054 R
ℓ = 0.8 R
ℓ = 1.0 R
ℓ = 1.5 R
ℓ = 2.108 R
ℓ = 2.33 R
ℓ = 3.33 R
ℓ = 5 R

X-axis: $\dfrac{S}{R}$ SIN ρ (NORMALIZED DISTANCE FROM CENTER OF PATROL PATH)

Y-axis: PROBABILITY OF DETECTION

We shall let $\overline{p}$ denote the "average probability of detection for effective patrol length L". For a uniform distribution of targets over A, this is the unweighted average over all p values with s ranging over the interval of values from zero to L/2. Then the formula for $\overline{p}$ is given by:

$$\overline{p} = \overline{p}(L, \ell, \rho) = \frac{\int_0^{\frac{L}{2}} p(s, \ell, \rho)\, ds}{\frac{L}{2}} \qquad (3.6)$$

where $p(s, \ell, \rho)$ is given by (3.5). Note that the average probability of detection is simply the proportion of the area of A that will be detected by the patrol at arbitrary (initial) point P.

4. OPTIMAL SOLUTION WITH EFFECT OF PARAMETERS

The problem of optimization is now tractable. In (3.6) fix v_t/v_p (i.e. fix ρ) and fix the average detection probability $\overline{p}$. For each v_t/v_p and each $\overline{p}$, a curve of ℓ vs L is generated. On this generated curve, we look for that value of the actual patrol length, ℓ, which maximizes the value of the effective patrol length, L. Let us denote this maximized value of L by "L MAXIMUM".

Figures 17, 18, 19 show the curves of actual patrol length vs effective patrol length for various average detection probabilities ranging over .75 (.05) 1.00 corresponding to $v_t/v_p = 1/3, 1, 3$, respectively.

EFFECT OF LOCATION OF TARGET BOUNDARY.

In Figures 17, 18, 19, the curves $L = \ell$, $L = \ell + 2R \sin \rho$, $L = $ MAXIMUM,
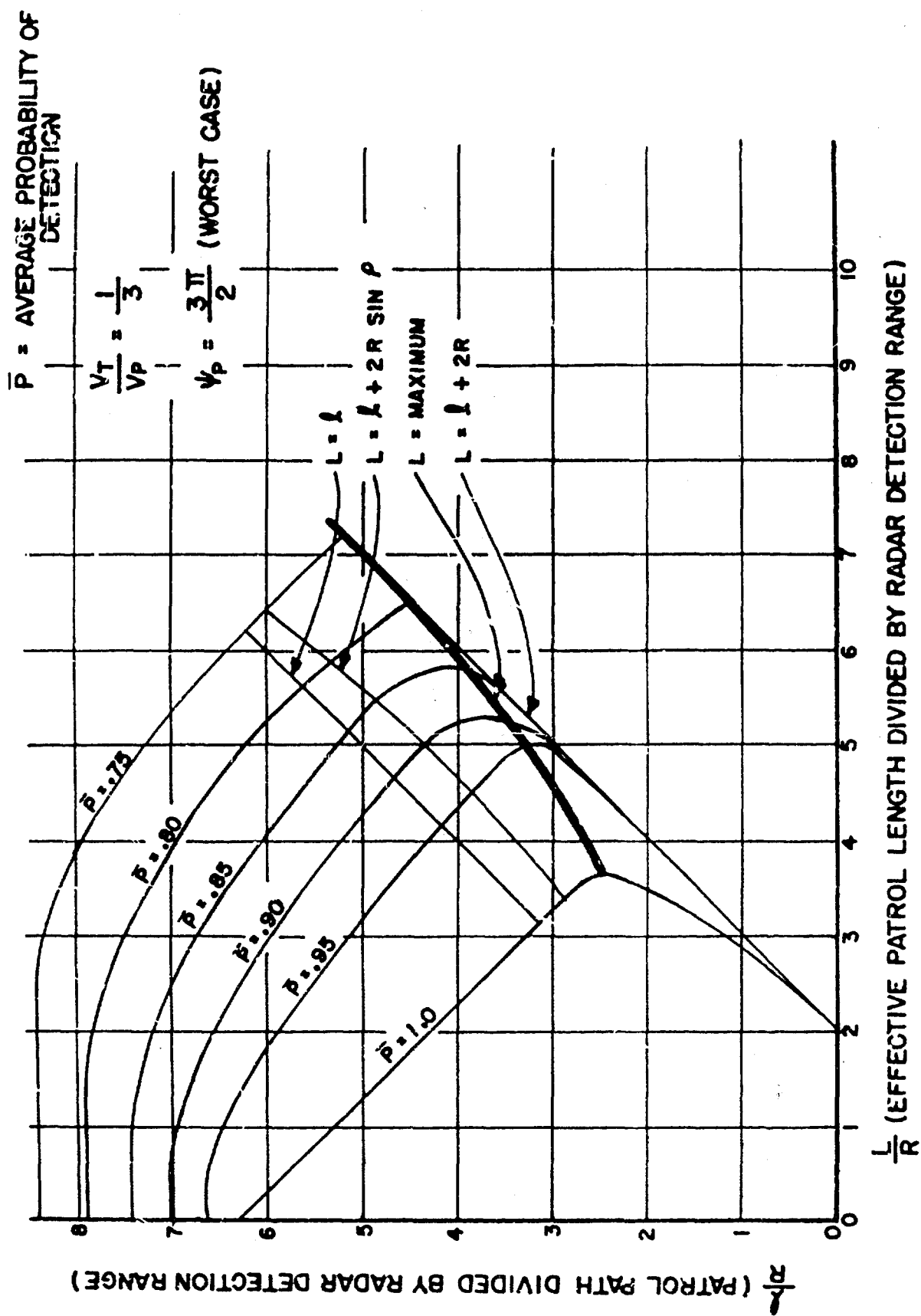
FIGURE 17. EFFECTIVE PATROL LENGTH vs. ACTUAL PATROL LENGTH FOR $v_t/v_p = 1/3$

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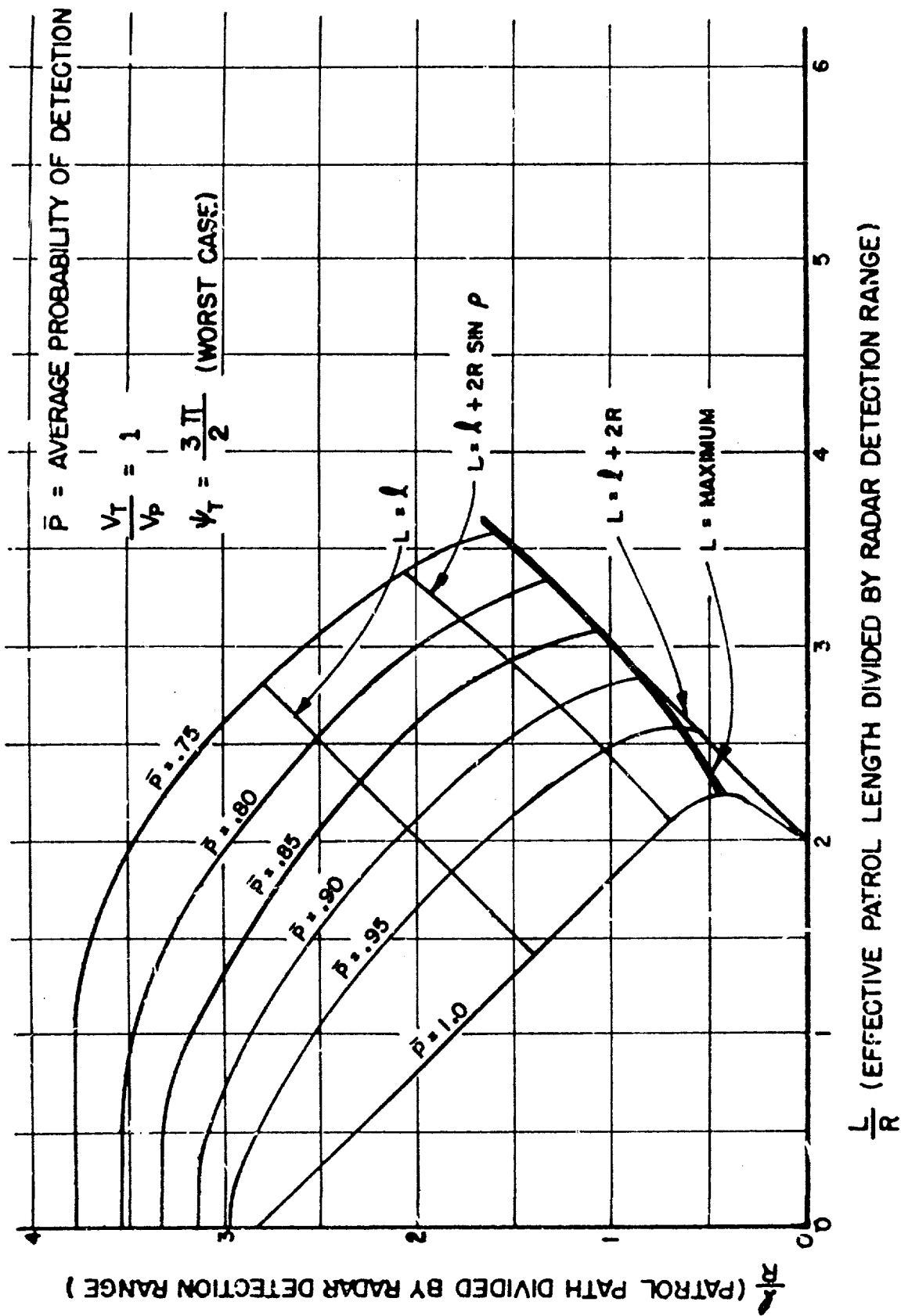
FIGURE 18. EFFECTIVE PATROL LENGTH vs. ACTUAL PATROL LENGTH FOR $v_t/v_p = 1$

The chart axes and labels:

- Vertical axis: $\frac{\ell}{R}$ (PATROL PATH DIVIDED BY RADAR DETECTION RANGE)
- Horizontal axis: $\frac{L}{R}$ (EFFECTIVE PATROL LENGTH DIVIDED BY RADAR DETECTION RANGE)

$\bar{P}$ = AVERAGE PROBABILITY OF DETECTION

$\frac{V_T}{V_P} = 1$

$\psi_T = \frac{3\pi}{2}$ (WORST CASE.)

$L = \ell$

$L = \ell + 2R \sin \rho$

$L = \ell + 2R$

$L = MAXIMUM$

$\bar{P} = .75$
$\bar{P} = .80$
$\bar{P} = .85$
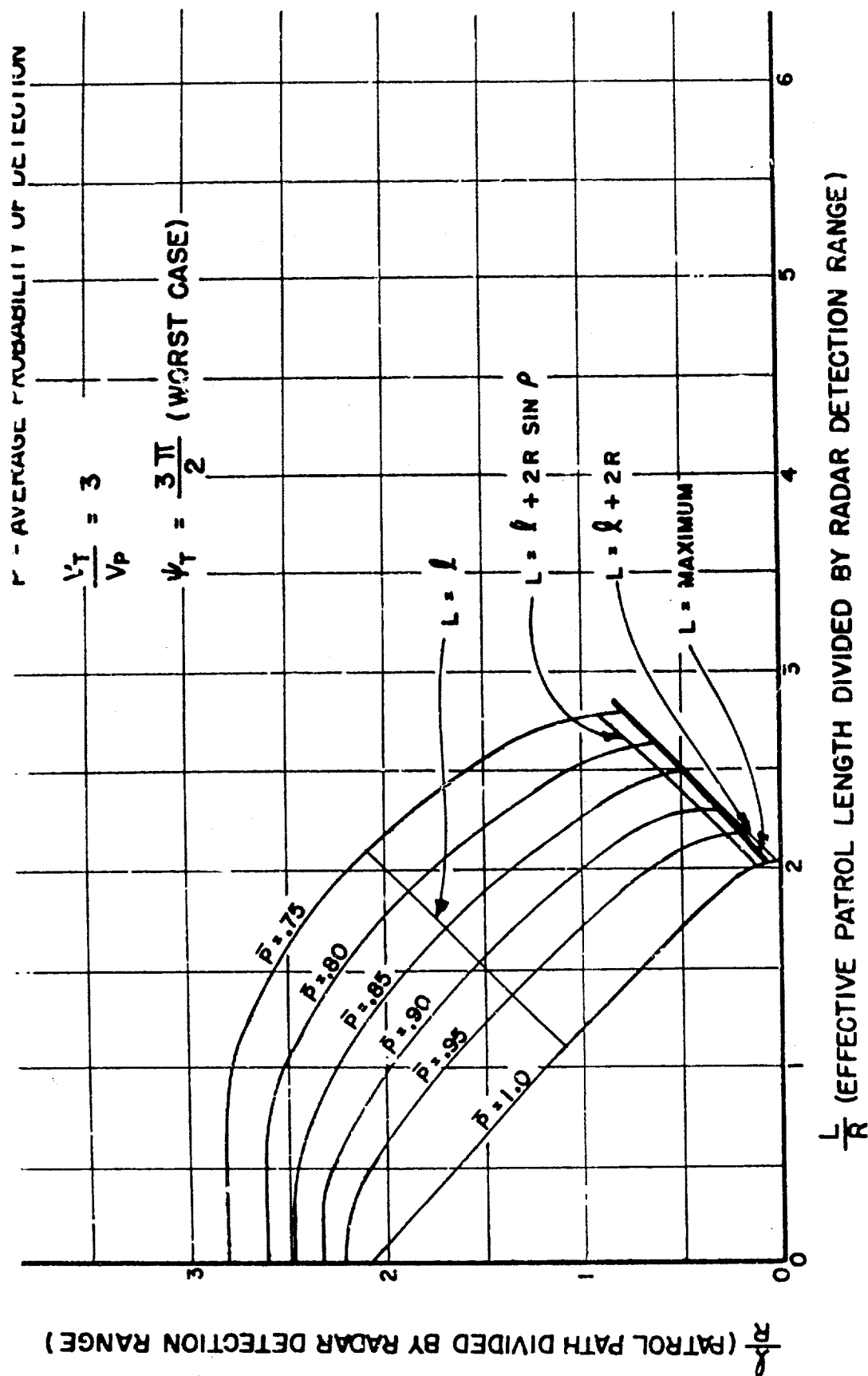$\bar{P} = .90$
$\bar{P} = .95$
$\bar{P} = 1.0$

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FIGURE 19. EFFECTIVE PATROL LENGTH vs. ACTUAL PATROL LENGTH FOR $v_t/v_p = 3$

$L = \ell + 2R$ indicated four possible locations of target boundaries, or equivalently, four possible choices of effective patrol length, L. It will be recalled that all targets approaching within these boundaries are to be policed by the patrol. Of these four possible choices, the best choice is $L = $ MAXIMUM; the worst choice is $L = \ell$, where ℓ is the actual patrol length.

For any given value of the speed ratio v_t/v_p, as $\bar{p}$ decreases, the curve $L = $ MAXIMUM approaches the curve $L = \ell + 2R$. In other words, for any fixed value of target-to-patrol speed ratio, when the required average probability of detection is low, then the maximum detection capability of the patrol is utilized where the maximum effective length includes the entire radar detection range at both ends of the actual patrol path.

EFFECT OF TARGET-TO-PATROL SPEED RATIO.

As v_t/v_p increases the curve $L = $ MAXIMUM again approaches the curve $L = \ell + 2R$. On the other hand, when the target moves at very great speeds relative to the patrol, the actual patrol length must be severely restricted to achieve high values for the average probability of detection, $\bar{p}$. For example, consider targets at a speed three times that of the patrol. Suppose we require that the patrol is to be capable of detecting all targets approaching within the target boundaries with at most this speed. Then the curve $\bar{p} = 1$ in Figure 19 $(v_t/v_p = 3)$ yields a value of the actual patrol length which is .08 times the radar detection range, a small value.

Figure 20 shows the maximum effective patrol length vs. target-to-patrol speed ratio for target headings $\psi_t = -90^\circ (\frac{3\pi}{2})$, -60°, -30°, where the average
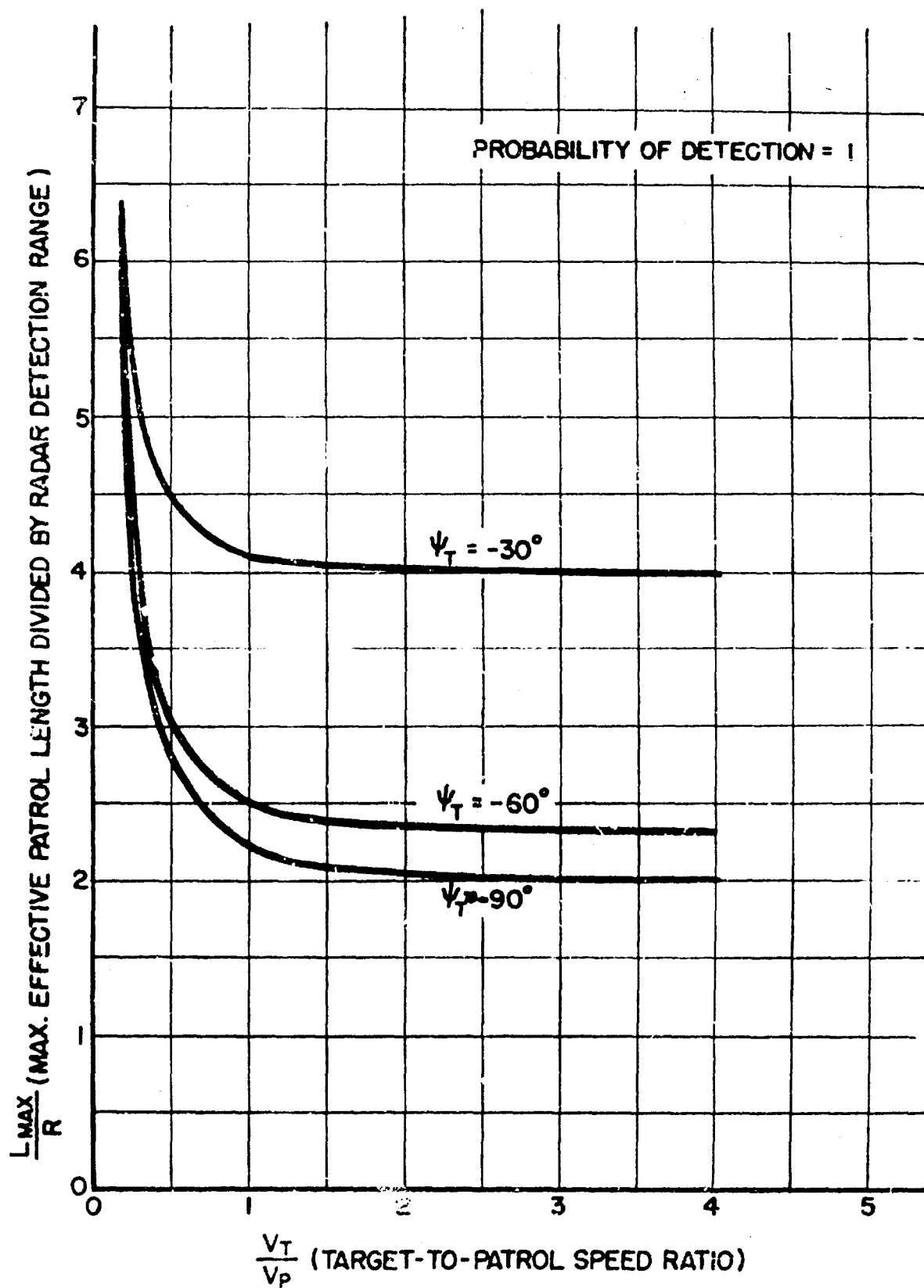
FIGURE 20. MAX. EFFECTIVE PATROL LENGTH vs. SPEED RATIO FOR

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probability of detection is equal to unity. It can be seen readily from Figure
20 that the target-to-patrol speed ratio has great effect on the maximum
effective patrol length. When the target speed is less than the patrol speed,
the maximum effective patrol length increases tremendously as the target-
to-patrol speed ratio decreases. On the other hand, when the target speed
is greater than the patrol speed, the maximum effective patrol length is
almost constant and asymptotically approaches twice the radar detection
range as the target-to-patrol speed ratio increases. Equivalently, the
actual patrol distance asymptotically approaches zero, as the target-to-
patrol speed ratio increases. For example, let the target speed be one-third
times and three times the patrol speed. The corresponding maximum effec-
tive patrol lengths are 3.6 times and 2.03 times the radar detection range.
Let the radar detection range be 200 nautical miles. The maximum effective
patrol length is 720 nautical miles for v_t/v_p equal to one-third; and it is 405
nautical miles for v_t/v_p equal to three. When the comparison is made on
the actual patrol length instead of the maximum effective patrol length for
these two speed ratios, the results are even more dramatic. For v_t/v_p equal
to one-third, the actual patrol length is 500 nautical miles; whereas, for v_t/v_p
equal to three, the actual patrol length is 16 nautical miles.

EFFECT OF TARGET HEADING.

As stated previously, the worst case is represented by the target heading
at -90° ($\frac{3\pi}{2}$). This curve in Figure 20 forms a lower bound on the maximum

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effective patrol length. When the target heading is changed from -90° to -60° with respect to the patrol path, the maximum effective patrol length increases slightly. However, when the target heading is between -60° to -30° with respect to the patrol path, an incremental change of target heading will result in greater incremental change of maximum effective patrol length. In fact, when the target heading approaches 0° or 180° with respect to the patrol path, the maximum effective patrol length rapidly becomes infinite.

EFFECT OF RADAR DETECTION RANGE.

The maximum effective patrol length and the corresponding actual patrol length are directly proportional to the radar detection range. For example, the maximum effective length is doubled, and the actual patrol length is doubled, when the radar detection range is doubled. Consequently the radar detection range greatly affects the ability of a patrol to detect a target. However, the radar detection range can not be increased indefinitely. It is limited by the line-of-sight distance, which in turn is affected by the curvature of the earth. This was discussed in Section 1 and shown in Figure 2.

PROBABILITY OF DETECTION AS A NON-INCREASING FUNCTION OF s.

For a given value of the actual patrol length ℓ and for a given horizontal target distance s from the center of the patrol path, there is a numerical value for the probability of detection, p, of the target. See Figures 15,

16,17. The maximum value of the detection probability occurs when the horizontal distance from the center approaches zero, i.e., when the target approaches the center of the patrol path. As s increases, the probability of detection decreases (or stays constant, and then eventually decreases). Thus, for minimum probability of detection the target should penetrate the patrol path at its ends.

SINGLE PATROL IN MULTIPLE LOOPS VS MULTIPLE PATROLS IN ONE LOOP.

Sometimes, the question arises as to whether two planes on two separate patrol loops are more, or less, effective than two planes on one patrol loop. For the back-and-forth pattern, the effective length of n patrols on n separate loops is nL, whereas the effective length of n patrols on one loop is given by (n -1) l + L. It follows that the effective length for separate loops is always longer than the effective length for one loop, whenever L is chosen longer than l . Thus, the logical answer to the question posed is that separate paths are always more effective than one path in regards to detection prob- ability, all other considerations being equal.

5. APPLICATIONS AND CONCLUSION

The primary function of an air patrol is to detect and identify an aerial penetrator. This must be done in sufficient time to alert the fighters so that they may intercept and destroy the penetrator before it reaches its intended target. In any typical operation of air defense patrol, a commmander may want answers to the following questions: for a given probability of detection,

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what is the minimum number of aircraft required? And how should these aircraft be flown to patrol a given boundary? Before these questions can be answered adequately, one must have prior knowledge of the penetrator's characteristics. The most important of these is the penetrator speed relative to the patrol speed. For a given penetrator there are various maximum speeds depending on the altitude at which the penetration is made. Generally, the speed is faster at high altitude than at low altitude. To cover a wide range of penetrator speed, in this example we choose the three target-to-patrol speed ratios of 1/3, 1, and 3.

AN EXAMPLE OF CONTINENTAL AIR DEFENSE.

Let's assume we are setting up an air defense patrol along the northern border of the United States, using only the back-and-forth pattern. The total distance between the States of Washington and Maine is approximately 3000 nautical miles. To obtain the minimum number of patrol planes required to cover the 3000 nautical miles distance, we simply divide the total distance by the maximum effective patrol length obtained from Figures 18, 19, 20. This maximum effective patrol length, L MAXIMUM, and its corresponding actual patrol length, l , for the three speed ratios, are tabulated in Table 1 for two average probabilities of detection of 1.0 and 0.9. The required minimum number of patrol planes, N, (for the three speed ratios and two average detection probabilities) is also tabulated in Table 1. These values are computed for a radar detection range equal to 200 nautical miles. A suggested air

AVERAGE DETECTION PROBABILITY = 1.0[*]

$\dfrac{v_t}{v_p}$	Maximum Effective Patrol Length (L) (Nautical Miles)	Actual Patrol Length (ℓ) (Nautical Miles)	Number of Aircraft Required, (N)
1/3	720	500	4. ?
1	450	92	6. 7
3	405	16	7. 4

[*] R assumed to be 200 nautical miles

AVERAGE DETECTION PROBABILITY = 0.9[*]

$\dfrac{v_t}{v_p}$	Maximum Effective Patrol Length (L) (Nautical Miles)	Actual Patrol Length (ℓ) (Nautical Miles)	Number of Aircraft Required, (N)
1/3	1060	720	2. 8
1	568	168	5. 3
3	460	60	6. 5

[*] R assumed to be 200 nautical miles

TABLE 1. MINIMUM NUMBER OF AIRCRAFT REQUIRED

defense patrol layout is shown in Figure 21.

Suppose one is interested in setting up an air defense patrol system which also includes the Atlantic and Pacific coasts of the United States. Instead of the 3000 nautical miles used, suppose the desired total patrol distance to be covered is actually 9000 miles. Then using the same procedure described above, we obtain the minimum numbers of aircraft required as three times the numbers given in Table 1.

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$$\bar{P} = 1$$
$$\frac{V_t}{V_P} = \frac{1}{3}$$
$$R = 200\text{NM}$$
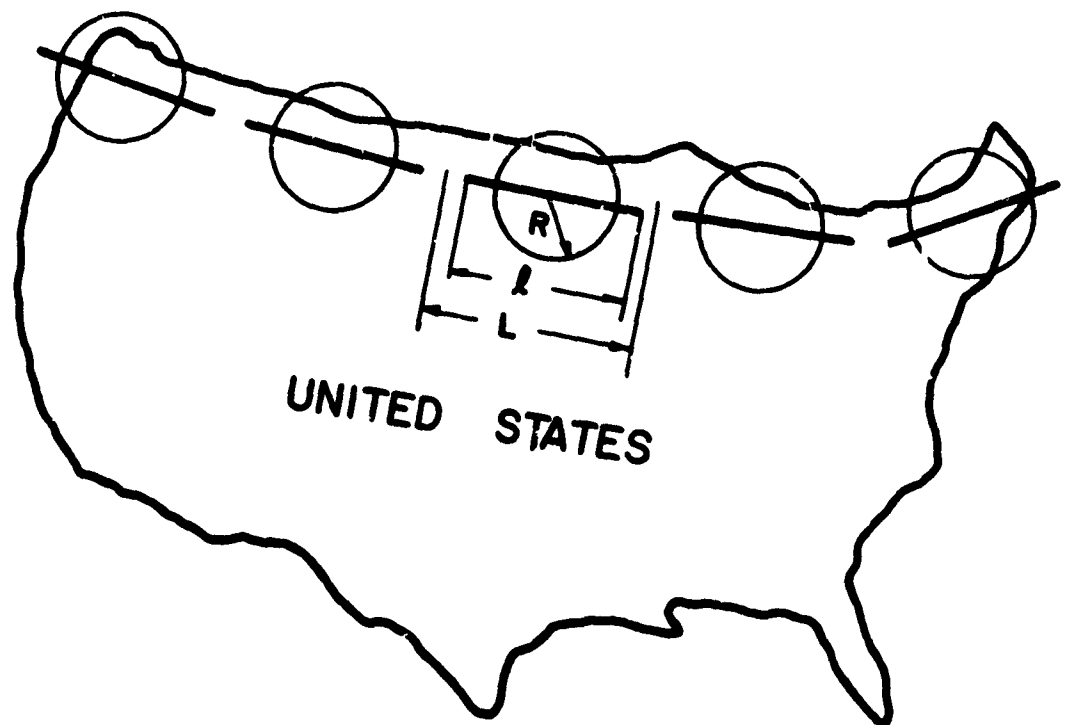$$L = 720\text{NM}$$
$$\ell = 500\text{NM}$$

UNITED STATES

FIGURE 21. SUGGESTED CONTINENTAL AIR DEFENSE PATROL LAYOU

Although three cases of target-to-patrol speed ratios are calculated, the case of target-to-patrol speed ratio of 3 may not lend itself to efficient high-speed air patrol operation. For this case, and for $\overline{p}= 1$, the maximum effective patrol length is 405 nautical miles, but the corresponding actual patrol length is only 16 nautical miles. For modern high-speed aircraft it may not be feasible to fly back and forth within such a short path. Analysis of Figures 17, 18, 19 indicates that we can obtain a longer length than 16 nautical miles by relaxing the requirement that the patrol be capable of detecting all such targets (i. e. , by choosing a value for the average probability of detection $\overline{p}$ which is less than unity), or by making the patrol fly faster. An equally reasonable alternative is to consider slower patrol aircraft which are capable of easily maneuvering short distances of the order of magnitude of 16 nautical miles or less. As can be seen by examining Figure 20, the reason why slower-moving aircraft also furnish a good solution is that once the target speed exceeds the patrol speed, further large increases in the value of the target-to-patrol speed ratio have relatively little effect on the maximum effective patrol length. For example, for $v_t/v_p = 3$, the maximum patrol length is 405 nautical miles; for $v_t/v_p = 10$, the maximum length is 403 nautical miles. Therefore, where the target-to-patrol speed ratio is greater than two, it may be quite practical to use much slower aircraft spaced apart at essentially twice the radar detection range, each plane hovering over some point. As indicated in Figure 20, the maximum effective length remains essentially the same for the slower aircraft as for the faster aircraft.

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If one wishes to reduce the required number of patrol planes for a given patrol distance, one can increase the patrol speed or increase the radar detection range while maintaining the same detection probability. Or one can maintain the same patrol speed and radar detection range but with a reduction of detection probability.

AN EXAMPLE OF COUNTER-INSURGENCY OPERATION.

Another possible application is for counter-insurgency operation in limited war. The air patrol may be used for the detection of low-altitude and low-speed aircrafts. Ground control intercept radar loses its effectiveness for low-altitude targets. Also it would be difficult to place ground radar installations along the patrol border which may be inaccessible, because of the terrain, except by air. This patrolled area may be infested with guerillas and may be subjected to constant insurgent harassment. On the other hand, air patrol will reduce this dependence of detection capability on target altitude, or on indigenous forces, or on terrain of the area to be patrolled. Furthermore, the aircraft that perform the patrol mission can be stationed at air bases safe from insurgent harassment.

By cruising at relatively high speed and high altitude the air patrol is not vulnerable to small-arms fire, which is typically encountered in COIN operation. The air patrol is highly mobile. It can be deployed on very short notice and can be withdrawn from one area and redeployed to another area very quickly.

But regardless of whether the situation is continental air defense or limited war air defense, the same method described in the above paragraph is used to determine the minimum number of patrol planes needed. The differences between these two situations is the len ;th of the patrol distances and the types of penetrators encountered.

For example, suppose the air patrol were to be used to patrol the South Vietnamese border that is adjacent to North Vietnam, Laos, and Cambodia. The patrol is to be against low-speed and low-altitude aircraft. The total length of the boundary is approximately 600 nautical miles. Based on the target-to-patrol speed ratio of 1 to 3, corresponding to average detection probability of unity, the maximum effective patrol length is 720 nautical miles for a radar detection range of 200 nautical miles. This means that only cne patrol aircraft is required to patrol this total region of South Vietnam. This one patrol will be capable of detecting all low-speed, low-altitude aircraft of speed ratio 1-to-3 or less. See Figure 22.

CONCLUSION.

In conclusion, we reiterate that numerical results have been obtained here for various probabilities of detection in the case where the target path is perpendicular to the patrol path (worst case), thus setting lower bounds on the maximum effective length for the back-and-forth patrol pattern. However, it will not be much more difficult to obtain numerical results for any arbitrary target heading angle. The main difference will be that one must contend with
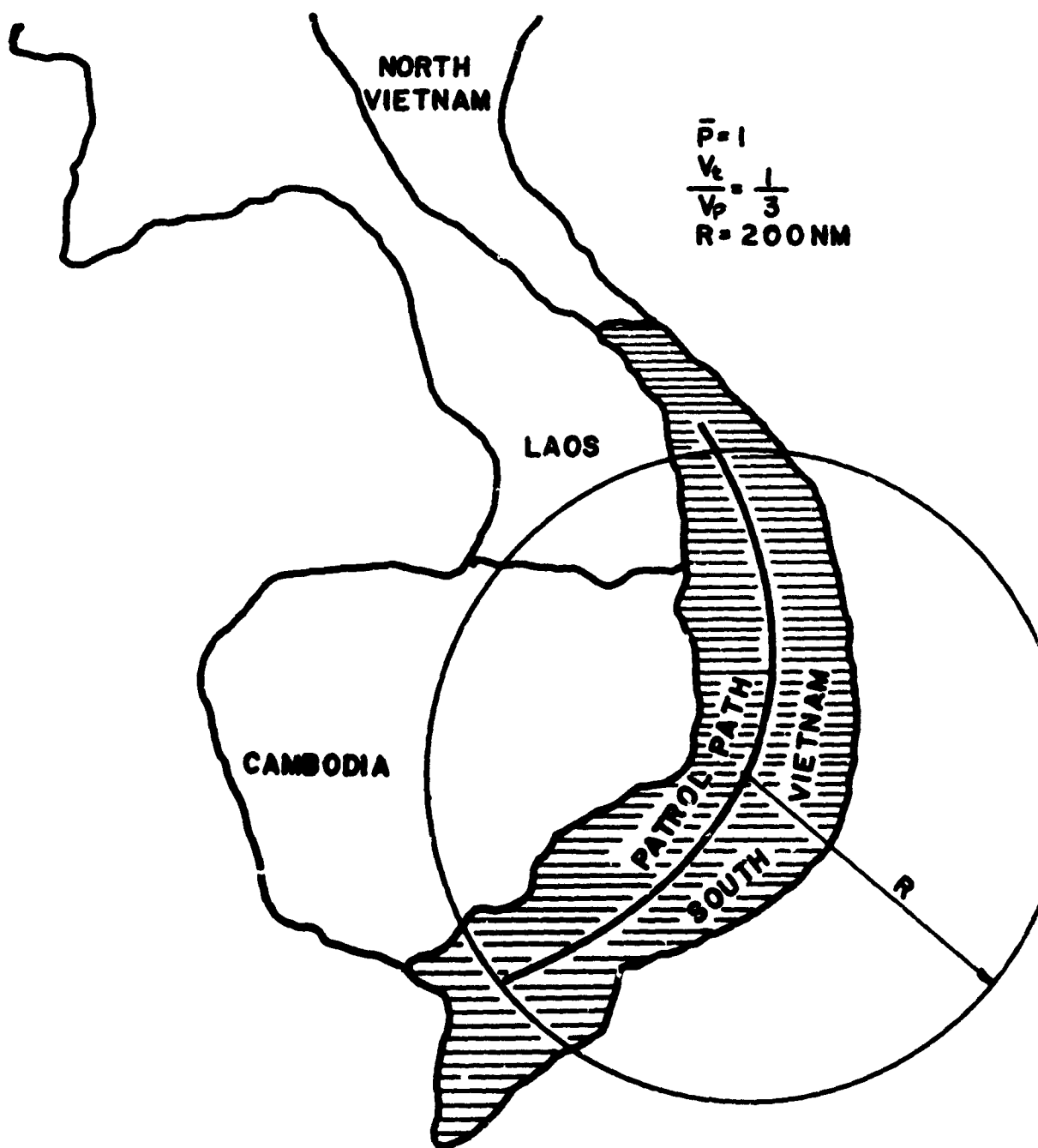
FIGURE 22. SUGGESTED PATROL PLAN FOR SOUTH VIETNAM

TWO relative velocity angles ρ_1, ρ_2 instead of the one relative velocity angle, ρ. For the rectangular or cross-over pattern, FOUR relative velocity angles must be considered. The effect on the detection probability resulting from a radar hole can readily be incorporated into our detection picture. One could also take into account more general radar patterns than circular; of immediate interest would be a radar detection area which is a sector of a circle.

A more serious complication arises when one attempts to generalize the definition of probability of detection. It is important to note that we have obtained in this paper the probability of instantaneously detecting a target. For some purposes one may prefer to know the proba⊦ ity of detecting a target for AT LEAST a MINUTES, where a is some number greater than zero. This is a more difficult problem whose consideration involves a detailed study of the other important measure of detection capability, detection time.

These concluding remarks are for the purpose of pointing out some possible directions in which the detection problem for repetitiouc patrol patterns can be generalized without undue complications in the theory. The authors have some ideas and partial results for a study of the detection time. They contemplate undertaking additional work to investigate these questions further.

REFERENCES

1. Koopman B. O., "The Theory of Search, I. Kinematic Bases".
    Journal of the Operations Research Society of America.
    4(1956) 324-346.

2. Lum, Gwon H., Poli, Corrado R., Lum, Mary D. Counter-Insurgency
    Study (Unclassified Title), ASD-TDR 63-292, Vol VI (SECRET)
    Patrols and Encounter, Aeronautical Systems Division,
    Wright-Patterson Air Force Base, Ohio, July, 1963.

3. Operations Research Group (ORG), A Theoretical Basis for Methods
    of Search and Screening, Summary Technical Report of Division 6,
    National Defense Research Committee, Volume 2B, Office of
    Scientific Research and Development, Washington, D. C., 1946.

APPENDIX

DERIVATION OF BOUNDARIES FOR THE DETECTION REGION
WITH PATROL INITIALLY AT POINT P

To determine the relative detection boundaries, in Figure 23, the target

proceeds from point T(x, y) at the same time the patrol proceeds from $P(x_p, y_p)$

with heading ψ_p. After a time duration of τ has passed, the patrol reaches

point $F(x_f, y_f)$ and the target at point $J(x_j, y_j)$ with $\vec{V}_r$ tangent to the radar circle.

This represents a limiting case for detection. We proceed to show that the

locus of initial target points T(x, y) satisfying this condition is a straight line

with slope m and intercept b, where m is the tangent of the relative velocity

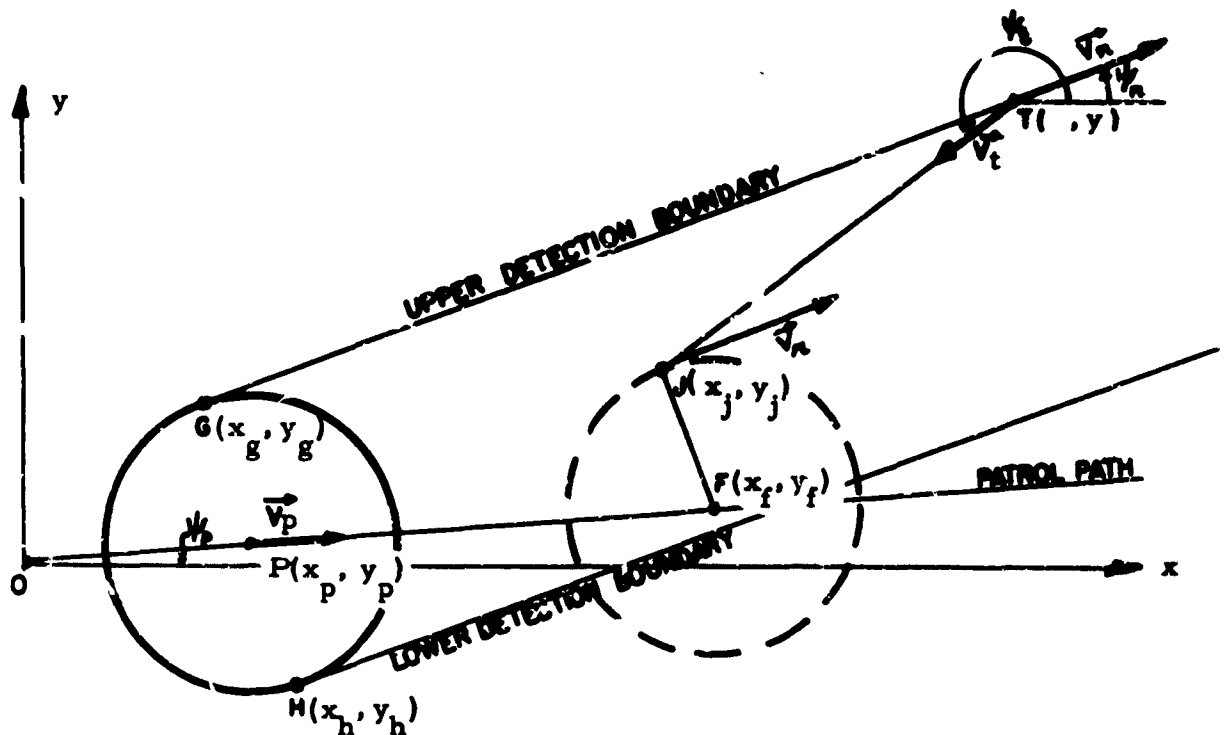heading ψ_r, and where b depends on m and $P(x_p, y_p)$.

FIGURE 23. DIAGRAM OF LIMITING CASE FOR DETECTION

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The general equations for the locus of points (x, y) are:

$$x_j = x + v_t \tau \cos \psi_t = x_p + v_p \tau \cos \psi_p + R \cos \lambda$$

$$(6.1)$$

$$y_j = y + v_t \tau \sin \psi_t = y_p + v_p \tau \sin \psi_p + R \sin \lambda$$

where $\lambda = \psi_r + \dfrac{\pi}{2}$ for $\psi_r \leq \dfrac{\pi}{2}$

$$= \psi_r - \dfrac{\pi}{2} \text{ for } \psi_r > \dfrac{\pi}{2}$$

Solving for τ in the first equation of (6.1),

$$\tau = \frac{x_p - x + R \cos \lambda}{v_t \cos \psi_t - v_p \cos \psi_p} \qquad (6.2)$$

Substituting τ into the second equation of (6.1), rearranging terms, and using (2.2), we obtain

$$y = mx + b, \qquad \text{a straight line} \qquad (6.3)$$

where $m = \dfrac{v_t \sin \psi_t - v_p \sin \psi_p}{v_t \cos \psi_t - v_p \cos \psi_p} = \tan \psi_r$

$$b = b(P) = y_p + R \sin \lambda - m \left[x_p + R \cos \lambda \right]$$

The straight line (6.3) is called the "upper detection boundary". It is tangent at the point G to the radar detection circle at $P(x_p, y_p)$

where the coordinates of G are $x_g = x_p + R \cos \lambda$, $y_g = y_p + R \sin \lambda$. Note that m depends ONLY on the relative velocity heading ψ_r, whereas b depends also on the initial patrol point $P(x_p, y_p)$. Of greater importance, however, is the fact that (6.3) does not directly depend on the target heading ψ_t except through ψ_r). This means we need consider a detection region dependent only on ψ_r and the initial position (point) P of the patrol.

Similarly, we can show that the "lower detection boundary" locus of target points is also a straight line with slope equal to that for the upper detection boundary and tangent at the point H to the detection circle at $P(x_p, y_p)$ where the coordinates of H are $x_h = x_p - R \cos \lambda$, $y_h = y_p - R \sin \lambda$.

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